Plan

1 Eksponentialfordeling
2 Simultant fordelte variabler
3 Uavhengighet og kovarians

Repetisjon:

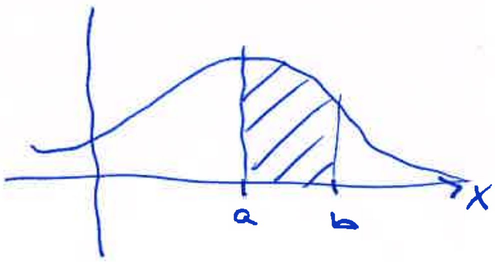
i) Kontinuerlig fordelte stokastiske variabler

X: Stokastisk variabel

X(S) Kontinuerlig mengde (intervall)

$$\boxed{P(a \le X \le b) = \int_a^b f(x)\,dx}$$

f(x): Sannsynlighets-tettheten til X



$$F(b) = P(X \le b) = \int_{-\infty}^{b} f(x)\,dx$$

kumulativ sannsynlighets-tetthet til X (cdf)

$$\boxed{F(x)' = f(x)}$$

i) $f(x) \ge 0$ for alle x

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$\mu = E(X)$$

$$Var(X) = E[(X-\mu)^2]$$

$$\sigma^2 = Var(X)$$

Regneregler for forventning og varians: Som før

$E(aX+b) = aE(X) + b$ $Var(aX+b) = a^2 \cdot Var(X)$ $Var(X) = E(X^2) - \mu^2$

ii) Normalfordeling:

$X \sim N(\mu, \sigma)$

normalfordelt

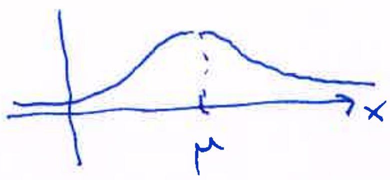
$\Downarrow$

$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

Standard normalfordelt

$X \sim N(\mu, \sigma)$

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



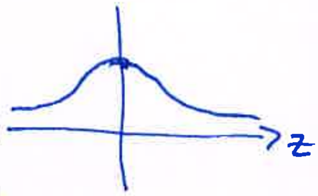
normalfordeling med parametre

$\mu = E(X)$

$\sigma^2 = Var(X)$

$Z \sim N(0,1)$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$



Standard-normalfordeling

$\mu = E(X) = 0$

$\sigma^2 = Var(X) = 1$

Eks: $X \sim N(\underset{\mu}{\underset{\shortparallel}{950}}, \underset{\sigma}{\underset{\shortparallel}{120}})$

$$p(X \geq 1100) = p\left(\frac{X-950}{120} \geq \frac{1100-950}{120}\right)$$

$$= p\left(Z \geq \frac{150}{120} = 1{,}25\right)$$

$$= 1 - p(Z \leq 1{,}25)$$

$$= 1 - G(1{,}25)$$

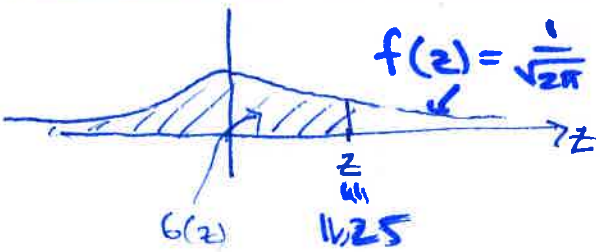
$$\simeq 1 - 0.895 \simeq \underline{\underline{0.105}}$$

$G(z) = p(Z \leq z)$

når $Z \sim N(0,1)$

cdf for standard normalf.

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

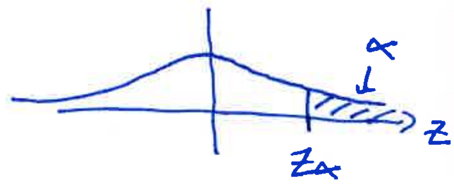


På kalkulator:

$G(1{,}25)$: 1,25 $\boxed{Z \rightleftarrows P}$

$\simeq 0{,}895$

Def: α-kvartilet z_α til std. normalfordeling



$p(Z \leq z_\alpha) = 1 - \alpha$

$G(z_\alpha) = 1 - \alpha$

På kalk:

0.95 $\boxed{INV}$ $\boxed{Z \rightleftarrows P}$ $\to 1{,}645$

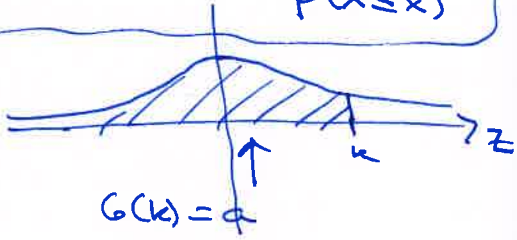
$\alpha = 0.05$: $1 - \alpha = 0.95$

$G(z_\alpha) = 0{,}95 \Rightarrow z_\alpha = G^{-1}(0{,}95) \simeq \underline{1{,}645}$

Husk: X kontinuerlig $\Downarrow$ $p(X = x) = 0$ $\qquad p(X<x) = p(X \le x)$

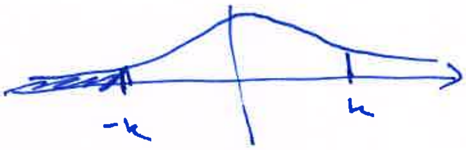
Oppgavesett 5

7. $Z \sim N(0,1) \qquad \Phi(k) = a$



a) $p(Z>k) = 1 - p(Z \le k) = \underline{\underline{1-a}}$

b) $p(Z<-k) = p(Z>k) = \underline{\underline{1-a}}$



c) $p(-k<Z<k) = 1 - 2\cdot(1-a)$

$= \underline{2a-1}$

eller

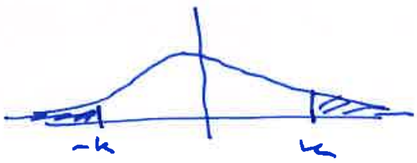
$p(-k<Z<k) = p(Z<k) - p(Z<-k)$

$p(a \le X \le b) = F(b) - F(a)$

$= \underset{\Phi(k)}{a} - \underset{\Phi(-k)}{(1-a)} = \underline{\underline{2a-1}}$

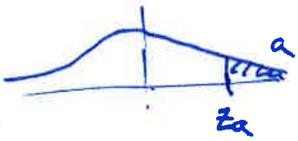
d) $p(|Z|>k) = p(Z>k) + p(Z<-k)$

$= (1-a) + (1-a) = \underline{\underline{2-2a}}$



9. Defn. av z_α:

$$\boxed{\begin{array}{c} p(Z \ge z_\alpha) = \alpha \\ \Updownarrow \\ p(Z \le z_\alpha) = 1-\alpha \end{array}}$$

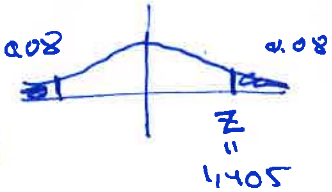


$\alpha = 0.08$:

$z_\alpha = z_{0.08} = \Phi^{-1}(0.92)$

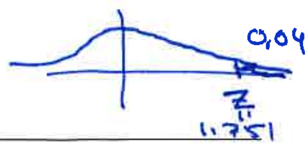
$\Phi^{-1}(0{,}08) \simeq -1{,}405$

0.92 $\boxed{\text{INV}}$ $\boxed{Z \rightleftarrows P}$ $\qquad z_{0.08} \simeq \underline{\underline{1{,}405}}$



$z_{\alpha/2} = z_{0{,}04} = \Phi^{-1}(0{,}96)$

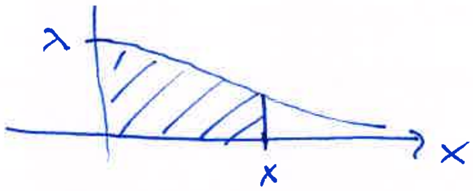
0.96 $\boxed{\text{INV}}$ $\boxed{Z \rightleftarrows P}$ $\qquad z_{0{,}04} \simeq \underline{\underline{1.751}}$



1) Eksponentialfordeling

$X \sim Exp(\lambda)$ $\qquad X(S) = [0, \to)$

$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$



$$\boxed{\begin{aligned} E(X) &= 1/\lambda \\ Var(X) &= 1/\lambda^2 \end{aligned}}$$

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \int_0^x \lambda e^{-\lambda x}\,dx \\ &= \left[-e^{-\lambda x}\right]_0^x \\ &= -e^{-\lambda x} - (-e^0) \end{aligned}$$

$$\boxed{F(x) = 1 - e^{-\lambda x}}$$

Man bruker denne fordelingen i en Poisson prosess, og

$\boxed{X = \text{ventetid til første forekomst.}}$

② Simultant fordelte stokastiske variabler

X og Y er to stokastiske variabler

Verdiene til X og Y avhenger av utfallet til et felles stokastisk forsøk.

Eks: Vi kaster to terninger

X = summen av terningene $\qquad X(S) = \{2, \dots, 12\}$

Y = differansen av terningene $\qquad Y(S) = \{0, \dots, 5\}$

(høyeste − laveste terning)

Y \ X	2	3	4	5	6	7	8	9	10	11	12	
0	1/36	0	1/36	0	1/36	0	1/36	0	1/36	0	1/36	6/36
1	0	2/36	0	2/36	0	2/36	0	2/36	0	2/36	0	10/36
2	0	0	2/36	0	2/36	0	2/36	0	2/36	0	0	8/36
3	0	0	0	2/36	0	2/36	0	2/36	0	0	0	6/36
4	0	0	0	0	2/36	0	2/36	0	0	0	0	4/36
5	0	0	0	0	0	2/36	0	0	0	0	0	2/36
	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36	1

$p(X=x, Y=y) = p(x,y)$

$p(2,0) = \frac{1}{36}$ $\qquad (1,1) \qquad p(2,1) = 0$

$p(3,0) = 0$ $\qquad p(3,1) = 2/36 \qquad (1,2) \quad (2,1)$

$p(4,0) = \frac{1}{36}$ $\qquad (2,2)$

$\vdots$

A: $X=3$ $\qquad A \cap B$: $X=3$ og $Y=0$

B: $Y=0$

Eks: $\boxed{p(X=3) = 2/36 \qquad p(X=3, Y=0) = 0 \qquad p(Y=0) = 6/36}$

$p(A) \cdot p(B) \neq p(A \cap B)$

Defn: X og Y er uavhengige hvis

$$p(X=x) \cdot p(Y=y) = p(X=x, Y=y)$$

for alle x og y.

Eks:

$X \backslash Y$	1	2	3	
1	0.1	0.2	0.2	0.5
2	0.1	0.1	0.3	0.5
	0.2	0.3	0.5	1

$X(S) = \{1, 2\}$
$Y(S) = \{1, 2, 3\}$

Simultane sannsynligheter

$p(x, y) = p(X=x, Y=y)$

$$p(1,2) = p(X=1, Y=2) = \underline{\underline{0.2}}$$

$p(x,y)$ (Simultane sannsynligheter) → $p_X(x)$ marginal X, → $p_Y(y)$ marginal Y

Marginale sannsynlighetsfordelinger:

X: $p(X=1) = 0.5$
$p(X=2) = 0.5$
$\quad 1$

$\cancel{p_X(x)}$

$$\boxed{\begin{aligned} p_X(1) &= 0.5 \\ p_X(2) &= 0.5 \end{aligned}}$$

$E(X) = 1 \cdot p_X(1) + 2 \cdot p_X(2)$
$= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \underline{\underline{1.5}} = \mu_X$

$E(X^2) = 1^2 \cdot p_X(1) + 2^2 \cdot p_X(2)$
$= 1 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} = 2.5$

$Var(X) = E(X^2) - 1.5^2 = 2.5 - 1.5^2$
$= 0.25 = 1/4 \qquad \sigma_X = 1/2 = 0.5$

Y: $p(Y=1) = 0.2$
$p(Y=2) = 0.3$
$p(Y=3) = 0.5$
$\quad 1$

$p_Y(1) = 0.2$
$p_Y(2) = 0.3$
$p_Y(3) = 0.5$

$E(Y) = 1 \cdot 0.2 + 2 \cdot 0.3 + 3 \cdot 0.5 = \underline{\underline{2.3}} = \mu_Y$

$E(Y^2) = 1^2 \cdot 0.2 + 2^2 \cdot 0.3 + 3^2 \cdot 0.5$
$= 0.2 + 1.2 + 4.5 = 5.9$

$Var(Y) = E(Y^2) - \mu_Y^2 = 5.9 - 2.3^2 = 0.61$

$\sigma_Y = \sqrt{0.61} \simeq \underline{\underline{0.78}}$

<u>Defn:</u> $Cov(X,Y) = E[(X-\mu_X)\cdot(Y-\mu_Y)]$

(<u>kovariansen</u> til X og Y)

<u>Merk:</u> i) $Cov(X,X) = E[(X-\mu_X)^2] = Var(X)$

ii) $Cov(Y,X) = Cov(X,Y)$

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E(XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y)$$
$$= E(XY) - \mu_X \cdot \underbrace{E(Y)}_{\mu_Y} - \cancel{\mu_Y \cdot \underbrace{E(X)}_{\mu_X}} \cancel{+\mu_X\mu_Y}$$
$$= E(XY) - \mu_X \cdot \mu_Y$$

iii) $$\boxed{\begin{aligned} Cov(X,Y) &= E(XY) - E(X)\cdot E(Y) \\ &= E(XY) - \mu_X\cdot\mu_Y \end{aligned}}$$

<u>1 eks:</u> $\mu_X = \underline{1.5}$ $\quad \mu_Y = \underline{2.3}$

X\Y	1	2	3
1	0.1	0.2	0.2
2	0.1	0.1	0.3

$$E(XY) = x_1 y_1 \cdot p(x_1,y_1) + x_1 y_2 \cdot p(x_1,y_2) + \dots$$
$$= 1\cdot1\cdot0.1 + 1\cdot2\cdot0.2 + 1\cdot3\cdot0.2 + 2\cdot1\cdot0.1 + 2\cdot2\cdot0.1 + 2\cdot3\cdot0.3$$
$$= 0.1 + 0.4 + 0.6 + 0.2 + 0.4 + 1.8 = \underline{3.5}$$

$$Cov(X,Y) = E(XY) - \mu_X\cdot\mu_Y = 3.5 - 1.5\cdot2.3 = \underline{\underline{0.05}}$$

<u>Resultat:</u> (1) Hvis X og Y er uavhengige, så er <u>$Cov(X,Y) = 0$</u>

(2) $Cov(X,Y)$ kan være negativ

Defn: **Korrelasjons koeffisient:**

$$\boxed{\rho = \rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}}$$

1 eks:

$$\rho = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} \simeq \frac{0{,}05}{0.5 \cdot 0.78} \simeq \underline{\underline{0.128}}$$

Resultat:

1. $-1 \leq \rho \leq 1$
2. $\rho > 0$: positiv samvariasjon
   $\rho < 0$: negativ — " —

Formler:

1) $E(aX + bY + c) = aE(X) + bE(Y) + c$

2) $Var(aX + bY + c) = a^2 \cdot Var(X) + b^2 \cdot Var(Y) + 2ab \cdot Cov(X,Y)$

Eks:

$$
\begin{aligned}
Var(X+Y) &= Cov(X+Y, X+Y) \\
&= Cov(X,X) + \underbrace{Cov(X,Y) + Cov(Y,X)} + Cov(Y,Y) \\
&= \underline{Var(X) + 2\,Cov(X,Y) + Var(Y)}
\end{aligned}
$$

Bevis: $-1 \leq \rho \leq 1$

a) $\mathrm{Var}\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) = \mathrm{Var}\left(\frac{1}{\sigma_X} \cdot X + \frac{1}{\sigma_Y} Y\right)$

$= \frac{1}{\sigma_X^2} \cdot \mathrm{Var}(X) + \frac{1}{\sigma_Y^2} \mathrm{Var}(Y) + 2 \cdot \frac{1}{\sigma_X} \cdot \frac{1}{\sigma_Y} \cdot \mathrm{Cov}(X,Y)$

$= \frac{\sigma_X^2}{\sigma_X^2} + \frac{\sigma_Y^2}{\sigma_Y^2} + 2 \cdot \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = 1 + 1 + 2\rho = 2 + 2\rho$

Sidan $\mathrm{Var}\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) \geq 0$, er $2 + 2\rho \geq 0$

$2\rho \geq -2$

$\rho \geq -1$

b) $\mathrm{Var}\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) = \frac{1}{\sigma_X^2} \mathrm{Var}(X) + \frac{1}{\sigma_Y^2} \mathrm{Var}(Y) - 2 \cdot \frac{1}{\sigma_X} \cdot \frac{1}{\sigma_Y} \mathrm{Cov}(X,Y)$

$= \frac{\sigma_X^2}{\sigma_X^2} + \frac{\sigma_Y^2}{\sigma_Y^2} - 2 \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = 2 - 2\rho$

Sidan $\mathrm{Var}\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) \geq 0$, er $2 - 2\rho \geq 0$

$-2\rho \geq -2$

$\rho \leq 1$

Konklusjon: $\left.\begin{array}{l} \rho \geq -1 \\ \rho \leq 1 \end{array}\right\}$ $-1 \leq \rho \leq 1$