Plan

1 Kontinuerlige stokastiske variabler
2 Uniform fordeling
3 Normalfordeling

Repetisjon:

Diskrete stokastiske variabler: $X(S)$ diskret mengde (endelig)

$p(X=x) = p(x)$

$E(X) = x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + \dots$ $\mu = E(X)$

$Var(X) = E[(X-\mu)^2]$ $\sigma^2 = Var(X)$

$= E(X^2) - \mu^2$ σ = std. avvik

$E(aX+b) = a \cdot E(X) + b$

$Var(aX+b) = a^2 \cdot Var(X)$

Binomisk: $Binom(n,p)$

$X(S) = \{0,1,\dots,n\}$

$p(X=x) = \binom{n}{x} p^x \cdot (1-p)^{n-x}$

X = antall ganger noe slår til
(n = antall gjentagelser)
(p = sannsynligheten for at det slår til)

$E(X) = np$ $Var(X) = np(1-p)$

Poisson: $Poisson(\lambda)$

$X(S) = \{0,1,2,\dots\}$

$p(X=x) = \frac{\lambda^x}{x!} e^{-\lambda}$

X = antall ganger en hendelse inntreffer i en gitt tidsperiode
(λ: gjennomsnittlig antall ganger det inntreffer)

$E(X) = \lambda$ $Var(X) = \lambda$

Standardisering:

X : stokastisk variabel $\rightsquigarrow$ $Z = \frac{X-\mu}{\sigma}$

$\mu = E(X)$
$\sigma^2 = Var(X)$

$\uparrow$ antall standardavvik fra μ til X

$E(Z) = E(\frac{1}{\sigma}(X-\mu))$
$= \frac{1}{\sigma}E(X-\mu)$
$= \frac{1}{\sigma}(E(X)-\mu)$
$= \underline{0}$

$Var(Z) = Var(\frac{1}{\sigma}(X-\mu))$
$= \frac{1}{\sigma^2}Var(X-\mu)$
$= \frac{1}{\sigma^2}\cdot Var(X)$
$= \frac{1}{\sigma^2}\cdot\sigma^2 = \underline{1}$

Chebyshev's ulikhet: $Z = \frac{X-\mu}{\sigma}$

$P(|Z|>k) \leq 1/k^2$ for $k>0$

Oppgaveark 4.

2. Fem sikre
Ti opps. med 50% sjanse

S = antall riktige på de ti oppgavene med 50% sjanse

$S \sim Binom(n=10, p=1/2)$

$R = 5+S$ antall riktige

$X = (5+S)\cdot 3 + (10-S)\cdot(-1)$
$= 15+3S-10+S$
$= \underline{5+4S}$

a) $P(R=15) = P(S=10)$
$= \binom{10}{10}\cdot(1/2)^{10}\cdot(1/2)^0 = \underline{(1/2)^{10}}$

$P(R=14) = P(S=9)$
$= \binom{10}{9}\cdot(1/2)^9\cdot(1/2)^1 = \underline{10\cdot(1/2)^{10}}$

$P(R=13) = P(S=8)$
$= \binom{10}{8}\cdot(1/2)^8\cdot(1/2)^2 = \underline{45\cdot(1/2)^{10}}$

b) $E(R) = E(5+S) = 5+E(S) = 5+10\cdot\frac{1}{2}$
$= \underline{10}$ $\quad \mu_R = 10$

$Var(R) = Var(5+S) = Var(S) = 10\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{5}{2} \Rightarrow \sigma_R = \sqrt{5/2}$

c) $E(X) = E(5+4S) = 5 + 4 \cdot E(S) = 5 + 4 \cdot 10 \cdot \frac{1}{2} = \underline{\underline{25}}$

$Var(X) = Var(4S+5) = 4^2 \cdot Var(S) = 16 \cdot 10 \cdot \frac{1}{2} \cdot \frac{1}{2} = \underline{40}$

$\Rightarrow \sigma_x = \underline{\underline{\sqrt{40}}}$

$\underline{|z| \leq 1}$: $X \in \left(25 - \sqrt{40},\ 25 + \sqrt{40}\right) \simeq (19, 31)$

d) $p(X \geq 37) = p(S \geq 8)$

$= p(S=8) + p(S=9) + p(S=10)$

$= p(R=13) + p(R=14) + p(R=15)$

$= (1/2)^{10} + 10 \cdot (1/2)^{10} + 45 \cdot (1/2)^{10}$

$= \underline{\underline{56 \cdot (1/2)^{10}}}$

$X = 5 + 4S$

37

$37 - 5 = 4S$

$\frac{32}{4} = \frac{4S}{4}$

$\underline{S = 8}$

$\underline{5}$ b) $X \sim \text{Poisson}(\lambda = 30)$

$p(X=x) = \frac{30^x}{x!} e^{-30}$

$p(X > 2) = p(3) + p(4) + \dots$

$= 1 - (p(0) + p(1) + p(2))$

$= 1 - \left[\frac{30^0}{0!} e^{-30} + \frac{30^1}{1!} e^{-30} + \frac{30^2}{2!} e^{-30} \right]$

$= 1 - e^{-30}(1 + 30 + 450) = \underline{\underline{1 - 481 \cdot e^{-30}}}$

① Kontinuerlige fordelinger

X: kontinuerlig stokastisk variabel

når de mulige verdiene X(S) til X er en kontinuerlig mengde

Typisk eksempler:

$X(S) = [0, 10]$

$X(S) = [0, \rightarrow)$

$X(S) = (\leftarrow, \rightarrow)$

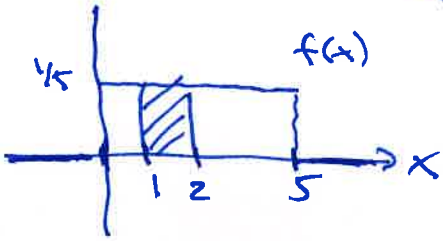
Sannsynlighetstetthet:

$f(x)$: kontinuerlig funksjon definert på $X(S)$.

Eks: $X(S) = [0,5]$

$$f(x) = \begin{cases} \frac{1}{5}, & x \in [0,5] \\ 0, & \text{ellers} \end{cases}$$

(tetthetsfn.)



$$p(1 \leq X \leq 2) = \int_1^2 f(x)\,dx = \left[\tfrac{1}{5}x\right]_1^2 = \tfrac{1}{5}\cdot 2 - \tfrac{1}{5}\cdot 1$$

$$= \frac{2-1}{5} = 1/5$$

Generelt om sannsynlighetstettheten $f(x)$:

$$\boxed{\begin{aligned} p(a \leq X \leq b) &= \int_a^b f(x)\,dx \\ p(X = a) &= 0 \quad \text{(punktsannsynlighet)} \end{aligned}}$$

$$F(x) = p(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

$F'(x) = f(x)$

Kumulativ sannsynlighetsfordeling $F(x)$

Oppsummering:

$f(x)$: tetthetsfunksjon

1) $p(a \le X \le b) = \int_a^b f(x)\,dx$

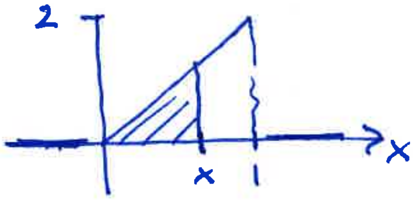
2) Krav: a) $f(x) \ge 0$
b) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$F(x)$: Kumulativ tetthetsfn.

i) $F(x) = \int_{-\infty}^{x} f(x)\,dx = p(X \le x)$

ii) $F'(x) = f(x)$

Eks: $f(x) = 2x$, $0 \le x \le 1$
$X(S) = [0,1]$



X: Stokastisk variabel med $X(S) = [0,1]$ og $f(x) = 2x$.

i) $f(x) \ge 0$ ✓

ii) $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 f(x)\,dx = 1$ ✓

$\int_0^1 f(x)\,dx = \int_0^1 2x\,dx = [x^2]_0^1 = 1 - 0 = \underline{1}$

$F(x) = p(X \le x) = \int_0^x 2x\,dx = [x^2]_0^x = x^2 - 0 = x^2$.

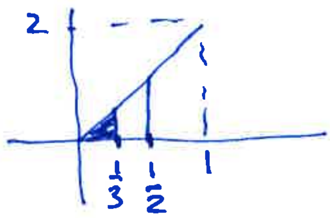
$\underline{F(x) = x^2}$, $0 \le x \le 1$

$F'(x) = (x^2)' = \underline{\underline{2x}}$

$p(X \le 1/2) = F(1/2) = \underline{\underline{1/4}}$

$p(X \le 1/3) = F(1/3) = \underline{\underline{1/9}}$

$p(1/3 \le X \le 1/2)$
$= F(1/2) - F(1/3) = \frac{1}{4} - \frac{1}{9}$
$= \frac{9-4}{36} = \underline{\underline{\frac{5}{36}}}$



Forventning og Varians: X kontinuerlig stokastisk var.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\, dx$$

Regneregler: Same som i det diskrete tilfellet

i) $E(aX+b) = a \cdot E(X) + b$

ii) $E(X+Y) = E(X) + E(Y)$

iii) $Var(X) \geq 0$

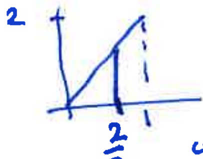
iv) $Var(aX+b) = a^2 \cdot Var(X)$

v) $Var(X) = E(X^2) - E(X)^2 = E(X^2) - \mu^2$

v) $$Var(X) = E[(X-\mu)^2] = E(X^2 - 2\mu X + \mu^2) = E(X^2) - 2\mu \cdot E(X) + \mu^2 = E(X^2) - 2\mu\cdot\mu + \mu^2 = \underline{E(X^2) - \mu^2}$$

Eks:

$f(x) = 2x, \quad 0 \leq x \leq 1$



$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_0^1 x \cdot 2x\, dx = \int_0^1 2x^2\, dx = \left[2 \cdot \tfrac{1}{3}x^3\right]_0^1 = \tfrac{2}{3}\cdot 1^3 - \tfrac{2}{3}\cdot 0^3 = \underline{\underline{2/3}}$$

$$E(X^2) = \int_0^1 x^2 \cdot f(x)\, dx = \int_0^1 2x^3\, dx = \left[\tfrac{2}{4}x^4\right]_0^1 = \tfrac{1}{2}\cdot 1^4 - \tfrac{1}{2}\cdot 0^4 = \underline{1/2}$$

$$Var(X) = E(X^2) - \mu^2 = 1/2 - (2/3)^2 = \frac{1\cdot 9}{2\cdot 9} - \frac{4\cdot 2}{9\cdot 2} = \underline{\underline{\frac{1}{18}}}$$

$\underline{\mu_X = \frac{2}{3}}$ $\quad \underline{\sigma_X = \sqrt{1/18}}$

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(2) Normalfordelingen

$X \sim N(\mu, \sigma)$

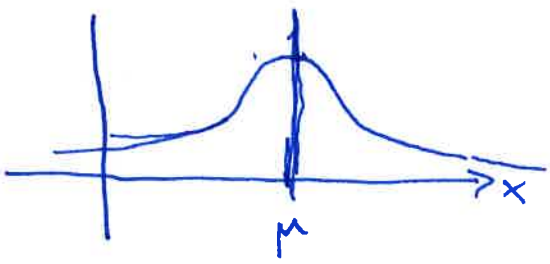
X er normalfordelt med parametre μ og σ

$E(X) = \mu$

$Var(X) = \sigma^2$

$X(S) = (-\infty, \infty)$

$$f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



(Gauss-kurve)

Standardisering:

$X \sim N(\mu, \sigma) \rightsquigarrow Z = \frac{X-\mu}{\sigma}$

standardisering

$E(Z) = 0$

$Var(Z) = 1$

Viktig:

X normalfordelt

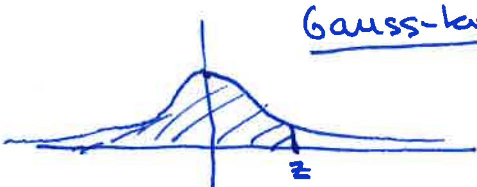
$\Downarrow$

$Z = \frac{X-\mu}{\sigma}$ normalfordelt

Standardnormalfordeling:

$Z \sim N(\mu = 0, \sigma = 1)$

Gauss-kurve



$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

$G(z) = P(Z \leq z)$

Eks. $G(1) = P(Z \leq 1) \simeq \underline{\underline{0.841}}$

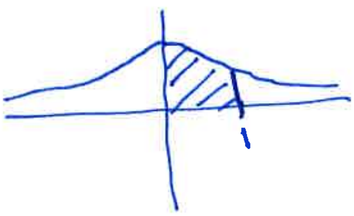
På kalkulator $\boxed{Z \rightleftarrows P}$

$1 \; \boxed{Z \rightleftarrows P}$

$P(0 \leq Z \leq 1) = G(1) - G(0)$

$\simeq 0.841 - 1/2$

$\simeq \underline{\underline{0.341}}$



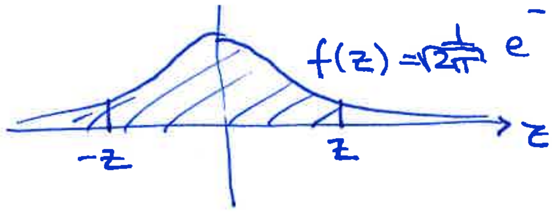
Standardnormalfordeling:

i) $X \sim N(\mu, \sigma) \Rightarrow Z = \dfrac{X-\mu}{\sigma}$

normalfordelt $\qquad \sim N(0,1)$

standard-normalfordelt

ii)



$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$

Gauss-kurven

- symmetrisk m/topp-pkt i $z=0$

$G(z) = P(Z \le z)$

$1 - G(-z) = G(z)$

$$\boxed{G(-z) = 1 - G(z)}$$

Eks: $G(-1) = 1 - G(1)$
$= 1 - 0.841$
$= \underline{0.159}$

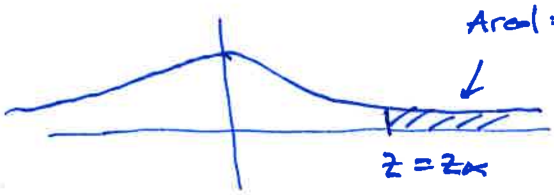
Eks: $X \sim N(12, 2) = N(\mu = 12, \sigma = 2)$

$P(X \ge 14) = P(Z \ge 1) = 1 - P(Z \le 1) = 1 - G(1)$
$= 1 - 0.841$
$= \underline{\underline{0.159}}$

$X = 14 \to Z = \frac{14-\mu}{\sigma} = \frac{14-12}{2} = 1$

Defn: α-kvantilet til Z: z_α

Areal $= \alpha \leftarrow P(Z > z_\alpha) = \alpha$



$\alpha = 0{,}05$:

$P(Z > z_{0,05}) = 0{,}05$

$P(Z \le z_{0,05}) = 1 - 0{,}05 = 0.95$

0,95 INV Z→P

$\underline{G(z) = 0.95}$

$z_{0,05} = \underline{\underline{1.645}}$

Forklaring:

$$X \sim N(\mu, \sigma) \implies Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

Anta at X er normalfordelt, og la $Z = \frac{X-\mu}{\sigma}$

Da har Z kumulativ fordelingsfunksjon: $F_Z(z)$ gitt ved

$$F_Z(z) = P(Z \le z) = P\left(\frac{X-\mu}{\sigma} \le z\right) = P(X - \mu \le \sigma z)$$

$$= P(X \le \mu + \sigma z) = F_X(\mu + \sigma z)$$

Dermed er tetthetsfunksjon $f_Z(z)$ gitt ved

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} F_X(\mu + \sigma z) = \underbrace{F_X'(\mu+\sigma z) \cdot (\mu + \sigma z)_z'}_{\text{kjerneregel for derivasjon}}$$

$$= f_X(\mu + \sigma z) \cdot \sigma$$

$$= \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{1}{2}\left(\frac{(\mu+\sigma z) - \mu}{\sigma}\right)^2} \cdot \sigma$$

tetthetsfunksjonen til X, som er normalfordelt $N(\mu, \sigma)$

$$= \frac{\cancel{\sigma}}{\sqrt{2\pi} \cdot \cancel{\sigma}} e^{-\frac{1}{2} z^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}$$

Siden $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}$ er tetthetsfunksjonen til ~~Standardnormalfordelingen~~ $N(0,1)$, er $Z \sim N(0,1)$ standardnormalfordelt.