Plan:

1. Kontinuerlige stokastiske variabler
2. Normalfordelingen
3. Eksponentiell fordelingen ← Neste gang.

Pensum:
[L] 4.3,
5.6–5.7

Repetisjon:

i) Diskrete fordelinger:
- binomisk
- Poisson
- geometrisk
- hypergeometrisk

se oversiktsark

Opps 5. Poisson-fordeling

$\lambda = \frac{30}{6} = 5$

X = antall oppringninger i et 10-min intervall

a) $P(X=2) = \frac{\lambda^2}{2!} e^{-\lambda} = \frac{5^2}{2!} e^{-5} = \frac{25}{2} e^{-5} \approx \underline{\underline{0.084}}$

ii) Chebyshevs ulikhet:

$$\boxed{P(|X-\mu| > k\sigma) \leq \frac{1}{k^2}}$$

når X har $E(X)=\mu$ og $Var(X)=\sigma^2$ og $k>0$

$|X-\mu| > k\sigma$

$\Updownarrow$

$\frac{|X-\mu|}{\sigma} > k$

$\boxed{|Z| > k}$

Standardisering:

X stok. var. $\Rightarrow Z = \frac{X-\mu}{\sigma}$ "z-score"

$\left.\begin{matrix}\mu = E(X) \\ \sigma^2 = Var(X)\end{matrix}\right\}$

antall std. avvik fra μ (ubenevnt)

Opps 6:

X
$\mu = 3$
$\sigma = 1.25$

$$\begin{array}{ccc} 1.75 & 3 & 4.25 \\ \| & \| & \| \\ \mu-\sigma & \mu & \mu+\sigma \end{array} \quad X$$

$$\begin{array}{ccc} -1 & 0 & 1 \end{array} \quad Z$$

$P(1 < X < 5)$

$\underline{X=1}$: $Z = \frac{1-3}{1.25} = \frac{-2}{1.25} = -1.6$

$\underline{X=5}$: $Z = \frac{5-3}{1.25} = \frac{2}{1.25} = 1.6$

(6) $P(1<X<5) = P(-1.6 < Z < 1.6)$

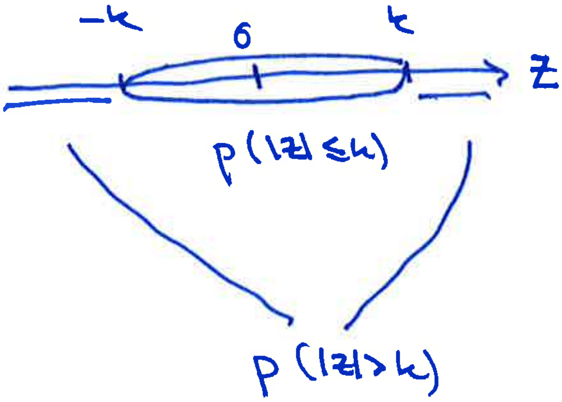
$= 1 - P(|Z|>1.6) \geq 1 - \frac{1}{1.6^2} \approx \underline{\underline{0.61}}$

Chebyshevs ulikhet:

$P(|X-\mu| > k\sigma) \leq 1/k^2$

$P\left(\frac{|X-\mu|}{\sigma} > k\right) \leq 1/k^2$

$\boxed{P(|Z|>k) \leq 1/k^2}$



(1) Kontinuerlige fordelinger: X kontinuerlig stokastisk variabel

Mulige verdier for X er en kontinuerlig mengde

Typiske eksempler: $X(S) = [0,10]$

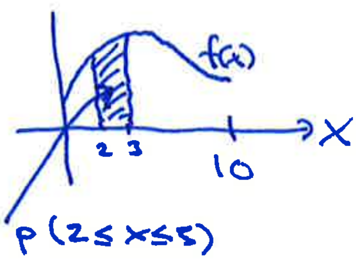
eller

$X(S) = (-\infty, \infty)$

Tetthetsfunksjonen til X:

kontinuerlig funksjon $f(x)$

$\boxed{\text{Defn: } P(a\leq X\leq b) = \int_a^b f(x)\,dx}$

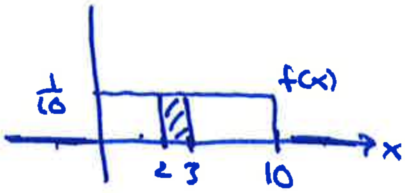


Tetthet: ~~"hvor sannsynlig et område er representert ved størrelse på f(x)"~~

$f(x)$ = hvor sannsynlig verdier omkring $X=x$ er

Eks: Uniform sannsynlighet

$f(x) = \frac{1}{10}, \quad 0 \le x \le 10$



$p(2 \le X \le 3) = \int_2^3 f(x)\,dx$

$= \int_2^3 \frac{1}{10}\,dx = \left[\frac{1}{10}x\right]_2^3 = \frac{3}{10} - \frac{2}{10} = \frac{1}{10} = \underline{\underline{0.10}}$

$p(X=3) = \int_3^3 f(x)\,dx = \underline{\underline{0}}$ ← Punktsannsynlighet er null for alle kontinuerlige variable

Husk: $p(a \le X \le b) = \int_a^b f(x)\,dx$

Generelle krav til tetthetsfunksjoner:

$$\boxed{\begin{array}{l} \text{i) } f(x) \ge 0 \text{ for alle } x \\ \text{ii) } \int_{X(S)} f(x)\,dx = 1 \end{array}}$$

X kontinuerlig stokastisk var.

f(x) tetthetsfunksjon til X (sannsynlighetsfordelingen til X)

Kumulative sannsynlighetsfordeling:

$F(x) = p(X \le x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} f(t)\,dt$ ← Dette betyr at $F'(x) = f(x)$!

Eks: $f(x) = \frac{1}{10}, \quad 0 \le x \le 10$

$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^{x} \frac{1}{10}\,dx = \left[\frac{1}{10}x\right]_0^x = \frac{1}{10}x - 0 = \underline{\underline{\frac{1}{10}x}}$

Svar: $F(x) = \frac{1}{10}x$ $\qquad F(3) = \frac{3}{10} = \underline{0.30}$

Forventning og varians for kontinuerlige stokastiske variabler

$$E(X) = \int_{X(S)} x \cdot f(x)\,dx$$

diskret:

$$E(X) = \sum_{X(S)} x \cdot p(x)$$

~~Var(X) =~~

$$E[g(X)] = \int_{X(S)} g(x) \cdot f(x)\,dx$$

$$E[g(X)] = \sum_{X(S)} g(x) \cdot p(x)$$

$$Var(X) = E[(X-\mu)^2] \quad \text{der } \mu = E(X)$$

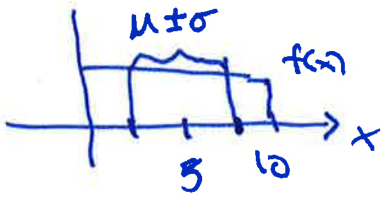
$$\boxed{\begin{array}{l} \text{Regneregler: } \textcolor{red}{(\text{som før!})} \\ \text{i)}\quad E(aX+b) = a \cdot E(X) + b \\ \text{ii)}\quad Var(aX+b) = a^2 \cdot Var(X) \\ \text{iii)}\quad Var(X) = E(X^2) - \mu^2 \end{array}}$$

Skrivemåte:

$\mu = E(X)$

$\sigma^2 = Var(X)$

$\sigma = \sqrt{Var(X)}$



Eks: $f(x) = \frac{1}{10}, \quad 0 \le x \le 10$

$$\mu = E(X) = \int_0^{10} x \cdot f(x)\,dx = \int_0^{10} \frac{1}{10} \cdot x\,dx$$

$$= \frac{1}{10} \cdot \left[\frac{1}{2}x^2\right]_0^{10} = \frac{1}{10}(50 - 0) = \frac{50}{10} = \underline{\underline{5}}$$

$$E(X^2) = \int_0^{10} x^2 \cdot f(x)\,dx = \int_0^{10} \frac{1}{10}x^2\,dx = \left[\frac{1}{10} \cdot \frac{1}{3}x^3\right]_0^{10}$$

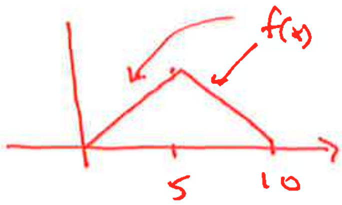
$$= \frac{1000}{30} - 0 = \underline{\frac{100}{3}} \Rightarrow Var(X) = E(X^2) - \mu^2$$

$$= \frac{100}{3} - 5^2 = \frac{100}{3} - \frac{75}{3} = \underline{\underline{\frac{25}{3}}}$$

$$\sigma = \sqrt{25/3} \simeq \underline{\underline{2.89}}$$

Eks: "Trekantfordeling" – ikke uniform

$$f(x) = \begin{cases} 0{,}04x, & 0 \le x \le 5 \\ 0{,}4 - 0.04x, & 5 \le x \le 10 \end{cases}$$



Regner ut:

i) $E(x) = \int_0^{10} x f(x)\,dx = \int_0^5 x \cdot 0.04x\,dx + \int_5^{10} x(0{,}4 - 0{,}04x)\,dx$

$$= \left[0{,}04 \cdot \tfrac{1}{3}x^3\right]_0^5 + \left[0.4 \cdot \tfrac{1}{2}x^2 - 0{,}04 \cdot \tfrac{1}{3}x^3\right]_5^{10}$$

$$= \frac{0{,}04 \cdot 125}{3} + 0.4(50 - 12.5) - 0{,}04 \cdot \left(\frac{1000}{3} - \frac{125}{3}\right)$$

$$= 0.04 \cdot \frac{125 \cdot 2 - 1000}{3} + 0.4 \cdot 37{,}5 = -10 + 15 = \underline{\underline{5}}$$

$\mu = 5$

ii) $E(x^2) = \int_0^{10} x^2 \cdot f(x)\,dx = \int_0^5 x^2 \cdot 0{,}04x\,dx + \int_5^{10} x^2(0{,}4 - 0{,}04x)\,dx$

$$= \left[0.04 \cdot \tfrac{1}{4}x^4\right]_0^5 + \left[0.4 \cdot \tfrac{1}{3}x^3 - 0{,}04 \cdot \tfrac{1}{4}x^4\right]_5^{10}$$

$$= 0.01 \cdot 625 + \frac{0.4}{3}(1000 - 125) - 0.01(10000 - 625) =$$

$$= 6{,}25 + \frac{350}{3} - 93.75 \cong 29{,}17$$

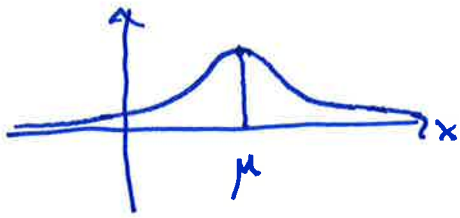
$\text{Var}(x) \cong 29.17 - \mu^2 \approx \underline{\underline{4.17}}$

$\sigma = 2{,}04$

(2) Normalfordelingen

μ, σ er gitte tall med $\sigma > 0$

$$f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}}$$



Symmetrisk om μ

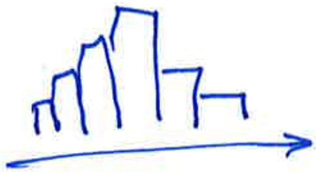
Gauss kurve

$\boxed{\text{Normalfordelingen } N(\mu,\sigma) \text{ har tetthetsfunksjon } f(x).}$

$X(S) = (-\infty, \infty)$

$$p(a \le x \le b) = \int_a^b f(x)\,dx \qquad F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$\boxed{\begin{aligned} E(x) &= \mu \\ Var(x) &= \sigma^2 \end{aligned}}$$



Standardnormalfordeling:

Hvis X er normalfordelt $N(\mu,\sigma)$, så kan vi se på $Z = \frac{X-\mu}{\sigma}$.

$$E(z) = E\left(\tfrac{1}{\sigma}\cdot(x-\mu)\right) = \tfrac{1}{\sigma}\left[E(x) - \mu\right] = 0 \quad \text{siden } E(x) = \mu$$

$$Var(z) = Var\left(\tfrac{1}{\sigma}(x-\mu)\right) = \tfrac{1}{\sigma^2} Var(x-\mu)$$

$$= \tfrac{1}{\sigma^2} \cdot Var(x) = \tfrac{1}{\sigma^2}\cdot\sigma^2 = 1$$

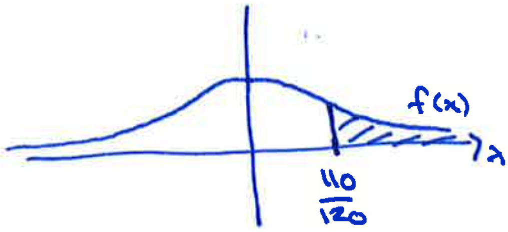
Resultat:

Hvis X er normalfordelt $N(\mu,\sigma)$, så $Z = \frac{X-\mu}{\sigma}$ normalfordelt $\underline{N(0,1)}$.

Standardnormalfordeling: $N(0,1)$

Normalfordeling med $\mu = 0$, $\sigma = 1$

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}x^2}$$



Eks:

Hvis X er $N(990,120)$: X er normalfordelt med $\mu = 990$ og $\sigma = 120$.

så er $Z = \frac{X-990}{120}$ $N(0,1)$: Z er standard normalfordelt

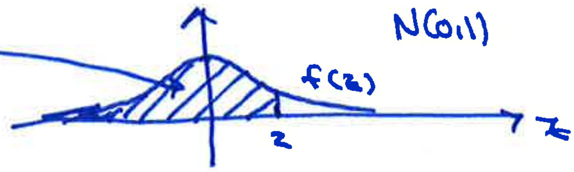
$$P(X > 1100) = 1 - P(X \leq 1100)$$
$$= 1 - P\left(Z \leq \tfrac{110}{120}\right)$$
$$= 1 - G\left(\tfrac{110}{120}\right)$$
$$\simeq 1 - 0.82$$
$$= \underline{\underline{0.18}}$$

Kalk: $110/120 = \boxed{Z \geq P}$

$$X \leq 1100$$
$$\frac{X-\mu}{\sigma} \leq \frac{1100-\mu}{\sigma}$$
$$Z \leq \frac{1100-990}{120}$$
$$Z \leq \frac{110}{120}$$

Defn: $G(z) = P(Z \leq z)$ i en standardnormalfordeling $Z \sim N(0,1)$.

$G(z)$ kumulative fordelingsfunksjon for $N(0,1)$.



Eks: X belastningen (i kg) på en bjelke

Anta X normalfordelt $N(990, 120)$

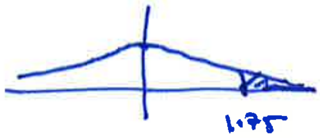
$$P(X>1200) = P(Z>1.75)$$
$$= 1 - P(Z \leq 1.75)$$
$$= 1 - G(1.75)$$
$$\approx 1 - 0.96 = \underline{\underline{0.04}} = 4\%$$

$X = 1200$

$\Updownarrow$

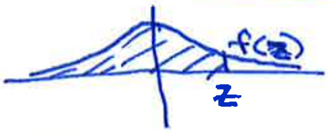
$$Z = \frac{1200 - \mu}{\sigma} = \frac{1200 - 990}{120} = \frac{210}{120} = \underline{1.75}$$

Z Std. normalf.



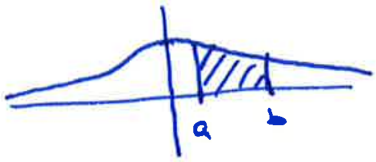
Oppsummering: Standard normalfordeling $N(0,1)$

Tetthetsfunksjon: $f(z) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



Kumulativ fordelingsfunksjon: $G(z) = \int_{-\infty}^{z} f(t)\,dt$ $\leftarrow$ På kalk: $\boxed{Z \rightleftarrows P}$

$$P(a \leq Z \leq b) = \int_a^b f(z)\,dz$$
$$= \underline{G(b) - G(a)}$$



$E(Z) = 0 \qquad Var(Z) = 1$

Viktig resultat:

X normalfordelt $N(\mu, \sigma)$ $\Rightarrow$ $Z = \frac{X-\mu}{\sigma}$ $\underline{\text{normalfordelt}}$ $N(0,1)$

Resultat:

X normalfordelt $N(\mu, \sigma)$ $\Rightarrow$ $Z = \frac{X-\mu}{\sigma}$ standard normalfordelt $N(0,1)$

Bevis:

$$G(z) = P(Z \le z) = P\left(\frac{X-\mu}{\sigma} \le z\right) = P(X - \mu \le \sigma z)$$

$$= P(X \le \mu + z\sigma) = \int_{-\infty}^{\mu+z\sigma} \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx$$

X er normalfordelt, så tetthetsfunksjonen er slik per defn.

$$G(z) = \int_{-\infty}^{\mu+z\sigma} f(x)\, dx$$

Kjerneregel for derivasjon med $u = \mu + z\sigma$ og $u' = \sigma$

$$G'(z) = f(\mu + z\sigma)\cdot\sigma$$

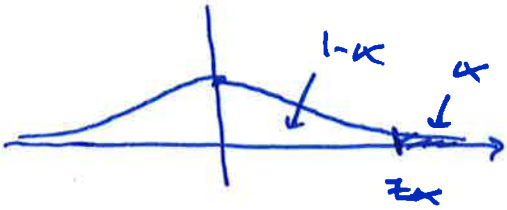
$$= \frac{1}{\sqrt{2\pi}\cdot\cancel{\sigma}} e^{-\frac{1}{2}\frac{(z\cancel{\sigma})^2}{\cancel{\sigma^2}}} \cdot \cancel{\sigma} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

Siden tetthetsfunksjonen til Z er $G'(z)$, og dette er lik tetthetsfunksjonen til en standard normalfordelt variabel, så er Z standard normalfordelt.

Kvantiler: for standard-normalfordeling

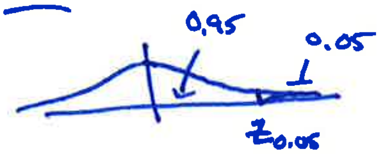
z_α kalles α-kvantilet og er definert som

$$P(Z > z_\alpha) = \alpha$$
$$\Updownarrow$$
$$P(Z \le z_\alpha) = 1 - \alpha$$
$$\Updownarrow$$
$$G(z_\alpha) = 1 - \alpha$$



$0 < \alpha < 1$

Eks: $\alpha = 0.05$



Regne ut $z_{0.05}$:

$G(z_{0.05}) = 0.95$ $z_{0.05} \simeq \underline{\underline{1.65}}$

Kalk:

0.95 $\boxed{\text{INV}}$ $\boxed{Z \rightleftarrows P}$ $\to$ $\underline{1.65}$

Noen regneregler for normalfordeling:

X er normalf. $N(\mu, \sigma)$

$$P(\mu - \sigma < X < \mu + \sigma) = P(-1 < Z < 1)$$
$$= G(1) - G(-1) \simeq \underline{\underline{0.68}} = 68\%$$

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(-2 < Z < 2)$$
$$= G(2) - G(-2) \simeq \underline{\underline{0.95}} = 95\%$$