Løsning: Eksamenssett III

BI

1.

a) $$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = \frac{1}{1/2}x^{1/2} + C = \underline{2\sqrt{x} + C}$$

3p. 3p.

b) $$\int \frac{9x-10}{x^2-3x+2}\,dx$$

Delbrøksoppspaltning:

$$x^2-3x+2 = (x-1)(x-2)$$

$$\frac{9x-10}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{x-2} \quad \Big| \cdot(x-1) \cdot(x-2)$$

$$9x-10 = A(x-2) + B(x-1)$$

$\underline{x=2}$: $8 = A\cdot 0 + B\cdot 1 \Rightarrow \underline{B=8}$

$\underline{x=1}$: $-1 = A\cdot(-1) + B\cdot 0 \Rightarrow \underline{A=1}$

$$\frac{9x-10}{x^2-3x+2} = \frac{1}{x-1} + \frac{8}{x-2}$$

3p.

$$= \int\left(\frac{1}{x-1} + \frac{8}{x-2}\right)dx$$

$$= \underline{\ln|x-1| + 8\ln|x-2| + C}$$

3p.

$$\left(= \ln|x-1| + \ln|x-2|^8 + C \right.$$
$$\left. = \ln\left|(x-1)(x-2)^8\right| + C \right)$$

c) $$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^u}{\cancel{\sqrt{x}}}\cdot 2\cancel{\sqrt{x}}\,du = \int 2e^u\,du$$

$$\boxed{\begin{gathered} u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\,dx \end{gathered}}$$

3p.

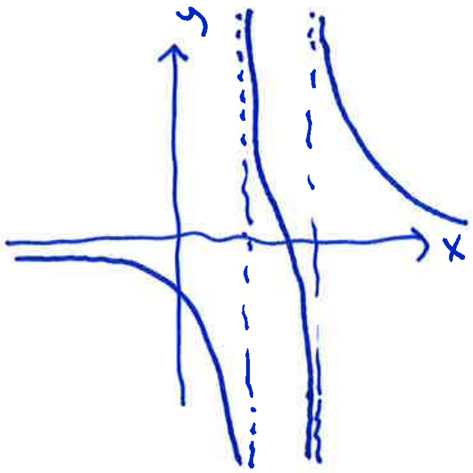
$$= 2e^u + C = \underline{2e^{\sqrt{x}} + C}$$

3p.

2. $f(x) = \dfrac{9x-10}{x^2-3x+2}$

a) Asymptoter: Vertikale er $x=1$ og $x=2$ (siden $9\cdot 1-10=-1\neq 0$, $9\cdot 2-10=8\neq 0$)

2p. Horisontale/skrå er $y=0$ siden nevner har størst grad.



2p.

$x\to\infty$: $y\to 0$, $y>0$

$x\to 2^+$: $y\to$ "$\frac{8}{0^+}$" $=+\infty$

$x\to 2^{\div}$: $y\to$ "$\frac{8}{0^{\div}}$" $=-\infty$

$x\to 1^+$: $y\to$ "$\frac{-1}{0^{\div}}$" $=+\infty$

$x\to 1^{\div}$: $y\to$ "$\frac{-1}{0^+}$" $=-\infty$

$x\to-\infty$: $y\to 0$, $y<0$

2p.

b) Vi regner ut $f''(x)$, og det lønner seg å bruke delbrøksoppspaltningen fra 1 b):

$$f(x) = \frac{1}{x-1} + \frac{8}{x-2} = (x-1)^{-1} + 8(x-2)^{-1}$$

$$f'(x) = -1\cdot(x-1)^{-2} - 8(x-2)^{-2}$$

$$f''(x) = 2(x-1)^{-3} + 16(x-2)^{-3} = \frac{2}{(x-1)^3} + \frac{16}{(x-2)^3}$$ 3p.

Vendepkt: $f''(x)=0$

$$\frac{2}{(x-1)^3} + \frac{16}{(x-2)^3} = 0 \quad \Big| \cdot (x-1)^3(x-2)^3$$

$$2(x-2)^3 + 16(x-1)^3 = 0$$

$$2(x-2)^3 = -16(x-1)^3 \quad \Big| \cdot \tfrac{1}{2}$$

$$(x-2)^3 = -8(x-1)^3 \quad \Big| \sqrt[3]{}$$

$$x-2 = \sqrt[3]{-8}\cdot(x-1) = -2(x-1)$$

$$x-2 = -2x+2$$

$$3x = 4$$

$x = \underline{\underline{4/3}}$ er eneste vendepkt. 3p.

$$\left(\begin{array}{l} \text{Hvis vi bruker } f(x) = \dfrac{9x-10}{x^2-3x+2} \text{ får vi} \\ f'(x) = \dfrac{-(x-2)^2 - 8(x-1)^2}{(x-1)^2(x-2)^2} = \dfrac{-9x^2+20x-12}{(x-1)^2(x-2)^2} \\ f''(x) = \dfrac{2(x-2)^3+16(x-1)^3}{(x-1)^3(x-2)^3} = \dfrac{18x^3-60x^2+72x-32}{(x-1)^3(x-2)^3} \end{array} \right)$$

c) Fra $f'(x) = -(x-1)^{-2} - 8(x-2)^{-2} = -\frac{1}{(x-1)^2} - \frac{8}{(x-2)^2}$

ser vi at $f'(x) < 0$ for alle $x \neq 1,2$. Dermed er f monotont <u>avtagende</u> i $(-\infty, 1)$ og $(1,2)$ og $(2,\infty)$, 2p. og f har en omvendt funksjon på hvert av intervallene. (se skisse i a) for å finne V_f)

Vi har:

$$\begin{array}{l} I = (-\infty, 1) \\ (1, 2) \\ (2, \infty) \end{array} \Rightarrow \boxed{\begin{array}{l} D_{f^{-1}} = V_f = (-\infty, 0) \\ D_{f^{-1}} = V_f = (-\infty, \infty) = \mathbb{R} \\ D_{f^{-1}} = V_f = (0, \infty) \end{array}} \leftarrow \text{2p.}$$

Vi må dermed velge <u>$I = (1,2)$</u> for at f^{-1} eksisterer med $D_{f^{-1}} = (-\infty, \infty) = \mathbb{R}$. 2p.

3.

a) $\left(\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 2 & 2 & 3 & 1 \\ 2 & 3 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\boxed{3p.}$

z fri

$2x + 2y + 3z = 1 \Rightarrow 2x = 1 - 5z \quad x = \frac{1}{2} - \frac{5}{2}z$

$y - z = 0 \Rightarrow y = z$

z fri

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 - 5/2\, z \\ z \\ z \end{pmatrix}, \quad z$ fri $\boxed{3p.}$

b) $|A| = \begin{vmatrix} a & 2 & 3 \\ 2 & a & 3 \\ 2 & 3 & a \end{vmatrix} = a \cdot (a^2 - 9) - 2(2a - 6) + 3(6 - 2a)$

$= a(a-3)(a+3) - 2(2a-6) - 3(2a-6)$

$= a(a-3)(a+3) - 5 \cdot 2(a-3)$

$= (a-3)\,(a(a+3) - 10)$

$= (a-3)\,(a^2 + 3a - 10)$

$= (a-3)\,(a-2)(a+5)$ $\boxed{3p.}$

eller

$(= a^3 - 19a + 30)$

$|A| = 0$: $a = 3$ eller $a = 2$ eller $a = -5$ $\boxed{3p.}$

c) $|A| \neq 0$: $a \neq 2, 3, -5$ én løsning

$|A| = 0$: $a = 2$ uendelig mange løsn. (fra a))

$\left.\begin{array}{l} a = 3 \\ a = -5 \end{array}\right\}$ Sjekk disse (se neste side)

$\boxed{2p.}$

$\underline{a=3}$:

$$\left(\begin{array}{ccc|c} 3 & 2 & 3 & 1 \\ 2 & 3 & 3 & 1 \\ 2 & 3 & 3 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 3 & 3 & 1 \\ 2 & 3 & 3 & 1 \\ 3 & 2 & 3 & 1 \end{array}\right) \begin{matrix} \\ -1 \\ -3/2 \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 2 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -5/2 & -3/2 & -1/2 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 3 & 3 & 1 \\ 0 & -5/2 & -3/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

uendelig mange løsn. (z fri)

$\boxed{2p.}$

$\underline{a=-5}$:

$$\left(\begin{array}{ccc|c} -5 & 2 & 3 & 1 \\ 2 & -5 & 3 & 1 \\ 2 & 3 & -5 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 3 & -5 & 1 \\ 2 & -5 & 3 & 1 \\ -5 & 2 & 3 & 1 \end{array}\right) \begin{matrix} \\ -1 \\ 5/2 \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 2 & 3 & -5 & 1 \\ 0 & -8 & 8 & 0 \\ 0 & 19/2 & -19/2 & 7/2 \end{array}\right) \frac{19}{16} \to \left(\begin{array}{ccc|c} 2 & 3 & -5 & 1 \\ 0 & -8 & 8 & 0 \\ 0 & 0 & 0 & 7/2 \end{array}\right)$$

ingen løsn.

$\boxed{2p.}$

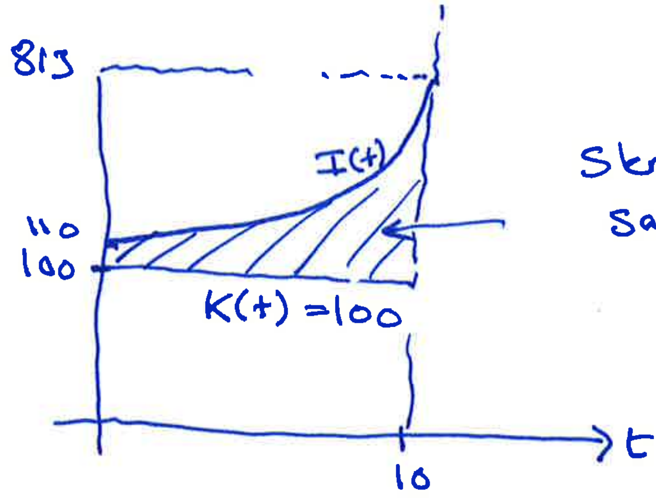
Uendelig mange løsn. for $\underline{a=2}$ og $\underline{a=3}$.

d) $$A^T \cdot A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 8 & 6 & 6 \\ 6 & 13 & 6 \\ 6 & 6 & 18 \end{pmatrix}}}$$

$\boxed{6p.}$

$I(10) = 110e^2 \simeq 812{,}8$

4.



Skravert areal = samlet driftsoverskudd $\boxed{3p.}$

Nåverdi: $r = 10\% = 0.10$

$$\int_0^{10} (110e^{t/5} - 100)e^{-rt}\,dt = \int 110e^{0.20t}e^{-0.10t} - 100e^{-0.10t}\,dt$$

$$= \int 110e^{0.10t} - 100e^{-0.10t}\,dt = \Big[110 \cdot 10e^{0.10t} - 100(-10)e^{-0.10t}\Big]_0^{10}$$

$$= 1100(e^1 - e^0) + 1000(e^{-1} - e^0) = 1100(e-1) + 1000\left(\frac{1}{e} - 1\right)$$

$\simeq 1.258$ mill. kr $\boxed{3p.}$

5. $f(x,y) = xy(x-y-1) = x^2y - xy^2 - xy$

a) $f'_x = 2xy - y^2 - y \quad = y(2x-y-1)$

$f'_y = x^2 - 2xy - x \quad = x(x-2y-1)$

eller $\boxed{3p.}$

Stasjonære pkt:

$y(2x-y-1) = 0$

$x(x-2y-1) = 0$

$y=0,\ x=0 \rightarrow (0,0)$

eller

$y=0,\ x=1 \rightarrow (1,0)$

eller

$x=0,\ y=-1 \rightarrow (0,-1)$ $\boxed{3p.}$

eller

$2x - y = 1 \quad \cdot 2$

$x - 2y = 1$

$-3x = -1 \rightarrow x = 1/3$

$y = -1/3$

$\rightarrow (1/3, -1/3)$

b) $H(f) = \begin{pmatrix} 2y & 2x-2y-1 \\ 2x-2y-1 & -2x \end{pmatrix}$ 2p.

$H(f)(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ $\det H = -1$ $\longrightarrow$ $(0,0)$ sadelpkt

$H(f)(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$ $\det H = -1$ $(1,0)$ sadelpkt

$H(f)(0,-1) = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$ $\det H = -1$ $(0,-1)$ sadelpkt 2p.

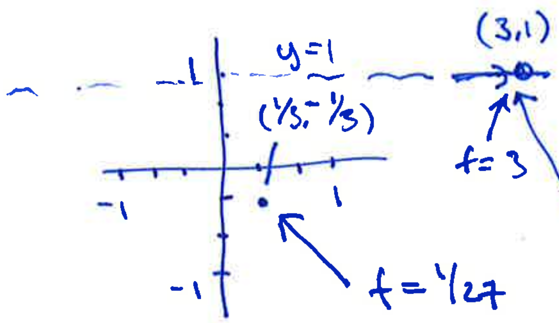
$H(f)(1/3,-1/3) = \begin{pmatrix} -2/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix}$ $\det H = 4/9 - 1/9 = 3/9 > 0$, $A = -2/3 < 0$ $\longrightarrow$ $(1/3,-1/3)$ lokalt maks

Globale maks/min:

Ingen globale min, siden ingen lokale min.

Mulig globalt maks: $f(1/3,-1/3) = -\frac{1}{9}\left(2/3-1\right) = \left(-\frac{1}{9}\right)\left(-\frac{1}{3}\right) = \frac{1}{27}$

Sjekker:



Hverken $x = 1/3$ eller $y = -1/3$ gir funksjonsverdier større enn $1/27$.

Prøver andre pkt lenger unna:

$y = 1$: $f(x,1) = x \cdot 1(x-1-1)$
$= x(x-2) = x^2 - 2x \to \infty$ når $x \to \infty$

For eks: $f(3,1) = 9 - 6 = 3 > \frac{1}{27}$

Konklusjon: Ingen globale maks eller min

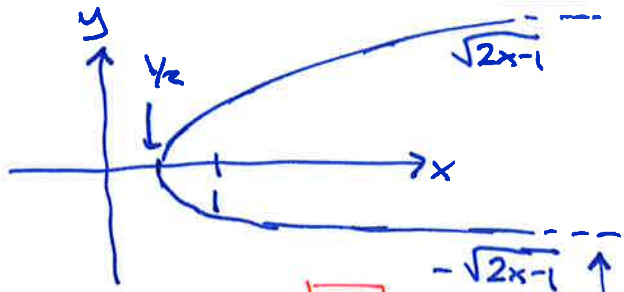
2p.

5. min $f = x^2 + y^2 - 4y$ når $2x - y^2 = 1$

a) $2x - y^2 = 1$

$y^2 = 2x - 1$

$y = \pm\sqrt{2x-1}$, $x \geq 1/2$



3p.

Ikke begrenset mengde siden det ikke er noe øvre grense for x (vi kan ha vilkårlig store verdier for x på kurven)

3p.

b) $L = x^2 + y^2 - 4y - \lambda(2x - y^2)$

$$\boxed{\begin{aligned} L'_x &= 2x - 2\lambda = 0 \\ L'_y &= 2y - 4 + \lambda \cdot 2y = 0 \\ & 2x - y^2 = 1 \end{aligned}}$$

Lagrange - betingelsene 3p.

$2x = 2\lambda \Rightarrow x = \lambda$

$2y - 4 + \lambda \cdot 2y = 0$

$2y - 4 + x \cdot 2y = 0$

$x = \dfrac{4 - 2y}{2y}$ $(y \neq 0)$

$2x - y^2 = 1$

$\cancel{2} \cdot \dfrac{4-2y}{\cancel{2}y} - y^2 = 1 \quad | \cdot y$

$4 - 2y - y^3 = y$

$0 = y^3 + 3y - 4$

$y^3 + 3y - 4 = 0$

Mulige heltallsløsn.

$y = \pm 1, \pm 2, \pm 4$

Prøver $y = 1$: $0 = 0$ ok.

$y^3 + 3y - 4 = (y-1)(y^2 + y + 4)$

kan finne denne direkte, eller ved polynomdiv.

$(y-1)(y^2 + y + 4) = 0$

$y = 1$ eller $y = \dfrac{-1 \pm \sqrt{1 - 4 \cdot 4}}{2}$ (ingen løsn)

$y = 1 \Rightarrow x = \dfrac{4-2}{2} = 1 \Rightarrow \lambda = 1 \rightarrow$ Konklusjon: $(x, y; \lambda) = (1, 1; 1)$ eneste pkt. som oppfyller Lagrange-betingelsene.

3p.

c) Lagrange-betingelsene: $(x,y;\lambda)=(1,1;1)$ $f=-2$

Degenerert bibetingelse: $g=2x-y^2$

$g'_x=2=0$
$g'_y=-2y=0$ } ingen slike pkt.

$\Downarrow$

Kun ett kandidatpkt $(1,1)$, ikke sikkert det gir min pga $2x-y^2=1$ ikke begrenset

$\boxed{2p.}$

Nivåkurver:

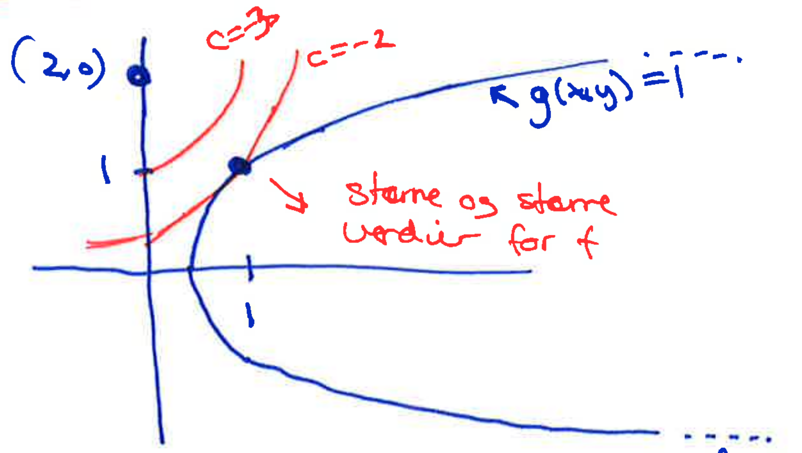
$f=x^2+y^2-4y=c$

$x^2+\underbrace{y^2-4y+4}_{(y-2)^2}=c+4$

$\boxed{2p.}$

$x^2+(y-2)^2=c+4$

sirkel m/ sentrum i $(0,2)$ og radius $\sqrt{c+4}$ $(c\geq -4)$



Radius $=1$: $c=-3$ dvs $f=-3$
Radius $=\sqrt{2}$: $c=-2$ dvs $f=-2$

Konklusjon

$f_{min}=-2$
i $(x,y)=(1,1)$ med $\lambda=1$

Vi ser fra tegningen at den minste f-verdien på kurven er $f=-2$ i $(1,1)$, når $x\to\infty$ så vil f få store verdier

$\boxed{2p.}$