Løsning: Eksamenssett II

1. (a) $\int_0^1 \frac{1}{(2-x)^2}\,dx = \int_2^1 \frac{1}{u^2}\cdot\frac{du}{-1} = \int_2^1 -\frac{1}{u^2}\,du$ 3p

$\boxed{\begin{aligned} u &= 2-x \\ du &= -dx \end{aligned}}$ $\quad u(1) = 2-1 = 1$, $\quad u(0) = 2-0 = 2$

$$= \left[\frac{1}{u}\right]_2^1 = \frac{1}{1} - \frac{1}{2} = 1 - 1/2 = \underline{1/2}$$ 3p

(b) $\int \sqrt{x}\cdot\ln x\,dx = \frac{2}{3}x^{3/2}\cdot\ln x - \int \frac{2}{3}x^{3/2}\cdot\frac{1}{x}\,dx$ 3p.

$\boxed{\begin{aligned} u &= \tfrac{2}{3}x^{3/2} & v &= \ln x \\ u' &= \sqrt{x} = x^{1/2} & v' &= 1/x \end{aligned}}$

$$= \frac{2}{3}x\sqrt{x}\ln x - \frac{2}{3}\int x^{1/2}\,dx = \frac{2}{3}x\sqrt{x}\ln x - \frac{2}{3}\cdot\frac{2}{3}x^{3/2} + C$$

$$= \underline{\frac{2}{3}x\sqrt{x}\ln x - \frac{4}{9}x\sqrt{x} + C}$$ 3p.

(c) $\int \frac{x}{(x+1)^2}\,dx = \int \frac{1}{x+1} + \frac{-1}{(x+1)^2}\,dx$ 3p.

$\frac{x}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} \quad \Big| \cdot (x+1)^2$

$x = A\cdot(x+1) + B$

$x = Ax + (A+B)$

$A = 1$, $A + B = 0$

$B = -1$

$\int \frac{-1}{(x+1)^2}\,dx = \int \frac{-1}{u^2}\,\frac{du}{1}$

$\boxed{\begin{aligned} u &= x+1 \\ du &= dx \end{aligned}}$

$= \int -\frac{1}{u^2}\,du = \frac{1}{u} + C$

$= \frac{1}{x+1} + C$

$$= \underline{\ln|x+1| + \frac{1}{x+1} + C}$$ 3p.

BI

2. $f(x) = \dfrac{8x}{(x+1)^2} - 1, \quad x \neq -1$

(a) $f(x) \geq 0$

$$\frac{8x}{(x+1)^2} - 1 \geq 0$$

$$\frac{8x}{(x+1)^2} - \frac{(x+1)^2}{(x+1)^2} \geq 0$$

$$\frac{8x - (x^2+2x+1)}{(x+1)^2} \geq 0$$

$$\frac{-x^2+6x-1}{(x+1)^2} \geq 0 \qquad \boxed{3p.}$$

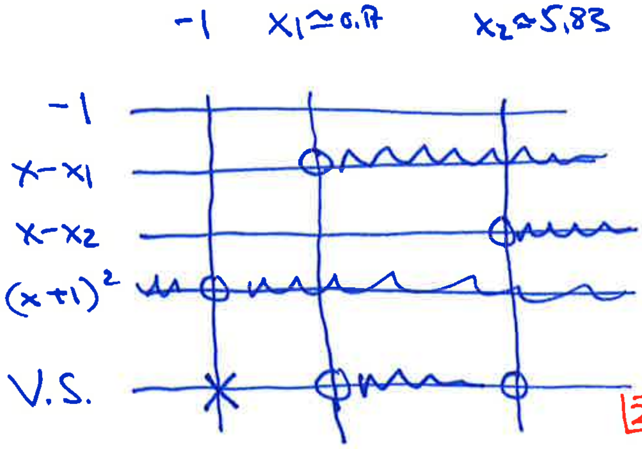
$$\frac{-(x-x_1)(x-x_2)}{(x+1)^2} \geq 0$$

$$-x^2+6x-1 = 0$$

$$x = \frac{-6 \pm \sqrt{36-4}}{-2} = 3 \pm \frac{\sqrt{32}}{2} = 3 \pm \frac{4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}$$

$$x_1 = 3 - 2\sqrt{2} \simeq 0.17$$

$$x_2 = 3 + 2\sqrt{2} \simeq 5.83$$

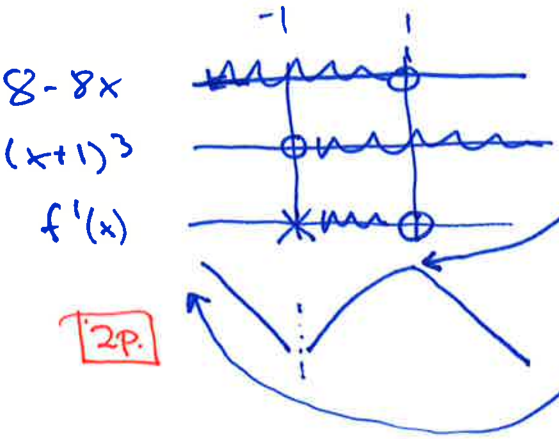


$\boxed{2p.}$

Løsning: $f(x) \geq 0$ for $x_1 \leq x \leq x_2$,
dvs $\underset{0.17}{3-2\sqrt{2}} \leq x \leq \underset{5.83}{3+2\sqrt{2}}$ $\boxed{1p.}$

b) $$f'(x) = \frac{8 \cdot (x+1)^2 - 8x \cdot 2 \cdot (x+1) \cdot 1}{(x+1)^4}$$

$$= \frac{\cancel{(x+1)} \cdot (8(x+1) - 16x)}{(x+1)^{\cancel{4}\,3}} = \frac{8x+8-16x}{(x+1)^3} = \frac{8-8x}{(x+1)^3} \qquad \boxed{3p.}$$



$\boxed{2p.}$

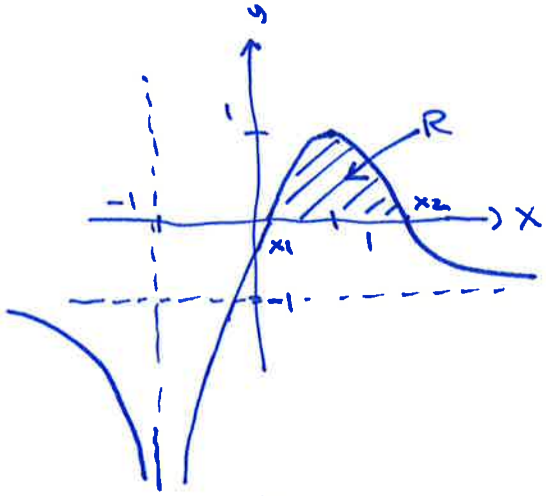
lok. maks: $x=1 \qquad f(1) = 1$

$x \to -\infty$: $f(x) = \dfrac{8x}{(x+1)^2} - 1 = \dfrac{8x}{x^2+2x+1} - 1 \to -1$

$\left(\text{siden } \dfrac{8x}{x^2+2x+1} \to 0 \right)$

Konklusjon: Maksimumsverdien til f er $f(x)=1$ i $x=1$ $\boxed{1p.}$

(c)



$\boxed{3p.}$ (med maksimum $f(1)=1$, nullpkt x_1, x_2)

$$A(R) = \int_{x_1}^{x_2} f(x) - 0 \, dx = \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{8x}{(x+1)^2} - 1 \, dx \qquad \boxed{2p.}$$

$$= \Big[8 \cdot \Big(\ln|x+1| + \frac{1}{x+1} \Big) - x \Big]_{3-2\sqrt{2}}^{3+2\sqrt{2}}$$

$$= \Big(8 \cdot \ln(4+2\sqrt{2}) + \frac{8}{4+2\sqrt{2}} \div (3+2\sqrt{2}) \Big) - \Big(8 \cdot \ln|4-2\sqrt{2}| + \frac{8}{4-2\sqrt{2}} \div (3-2\sqrt{2}) \Big)$$

$$= 8 \cdot \ln \frac{4+2\sqrt{2}}{4-2\sqrt{2}} + \frac{8}{4+2\sqrt{2}} - \frac{8}{4-2\sqrt{2}} - 4\sqrt{2} \simeq \underline{2.79} \qquad \boxed{1p.}$$

3. $A = \begin{pmatrix} 0 & a & 2 \\ a & 1 & a \\ 2 & a & 0 \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 1 \\ a \\ 1 \end{pmatrix}$

(a) $\underline{a=1}$:
$$\left(\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

trappeform $\boxed{3p.}$

uendelig mange løsn.
z fri variabel

$$\begin{aligned} x+y+z&=1 \\ y+2z&=1 \end{aligned} \quad \to \quad \begin{aligned} x &= 1-y-z = 1-(1-2z)-z = z \\ y &= 1-2z \\ z &= \text{fri} \end{aligned}$$

$\boxed{3p.}$ ✓

$(x,y,z) = \underline{(z, 1-2z, z)}$ med z fri

[redacted]

(b) $|A| = \begin{vmatrix} 0 & a & 2 \\ a & 1 & a \\ 2 & a & 0 \end{vmatrix} = 2(a^2-2) - a(0-2a)$

$= 2a^2 - 4 + 2a^2$

$= \underline{4a^2-4} = 4(a^2-1) = \underline{4(a-1)(a+1)}$ 3p.

$|A| = 0$: $\quad 4(a-1)(a+1) = 0$

$\underline{a = 1,\ a = -1}$ 3p.

(c) Fra teori: $|A| = 0$: ingen/uendelig mange løsn. 2p.
$|A| \neq 0$: én løsn.

$|A|=0$

$\underline{a=1}$: $(x,y,z) = (z, 1-2z, z)$ med z fri fra (a). 2p.
(uendelig mange løsn.)

$\underline{a=-1}$: $\left(\begin{array}{ccc|c} 0 & -1 & 2 & 1 \\ -1 & 1 & -1 & -1 \\ 2 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 1 & -1 & -1 \\ 0 & -1 & 2 & 1 \\ 2 & -1 & 0 & 1 \end{array}\right) 2$

$\to \left(\begin{array}{ccc|c} -1 & 1 & -1 & -1 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & -2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 1 & -1 & -1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$-x + y - z = -1$ (uendelig mange løsn.) 2p.
$-y + 2z = 1$
(z fri)

Konklusjon: Uendelig mange løsninger for $\underline{\underline{a = \pm 1}}$

(d) Fra teori: $|A^2| = 0$: ingen/uendelig mange løsn. 2p
$|A^2| \neq 0$: én løsn.
$(A^2 \cdot \underline{x} = \underline{b})$

$|A^2| = |A \cdot A| = |A| \cdot |A| = \left(4(a-1)(a+1)\right)^2 = 16(a-1)^2(a+1)^2$ 2p

$|A^2| = 0$ for $a = \pm 1$.

$\underline{a=1}$: $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \Rightarrow A^2 = A \cdot A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 3 & 1 \\ 3 & 3 & 3 \\ 1 & 3 & 5 \end{pmatrix}$

$A^2 \cdot \underline{x} = \underline{b}$:

$$\left(\begin{array}{ccc|c} 5 & 3 & 1 & 1 \\ 3 & 3 & 3 & 1 \\ 1 & 3 & 5 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 3 & 3 & 3 & 1 \\ 5 & 3 & 1 & 1 \end{array}\right) \begin{matrix} \\ -3 \\ -5 \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 0 & -6 & -12 & -2 \\ 0 & -12 & -24 & -4 \end{array}\right) \begin{matrix} \\ \\ -2 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 0 & -6 & -12 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

z fri
uendelig mange løsn.
1p.

$\underline{a=-1}$: $A^2 = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 \\ -1 & +1 & -1 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & -3 & 1 \\ -3 & 3 & -3 \\ 1 & -3 & 5 \end{pmatrix}$

$A^2 \cdot \underline{x} = \underline{b}$:

$$\left(\begin{array}{ccc|c} 5 & -3 & 1 & 1 \\ -3 & 3 & -3 & -1 \\ 1 & -3 & 5 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -3 & 5 & 1 \\ -3 & 3 & -3 & -1 \\ 5 & -3 & 1 & 1 \end{array}\right) \begin{matrix} \\ 3 \\ -5 \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 1 & -3 & 5 & 1 \\ 0 & -6 & 12 & 2 \\ 0 & 12 & -24 & -4 \end{array}\right) \begin{matrix} \\ \\ 2 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & -3 & 5 & 1 \\ 0 & -6 & 12 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

z fri
uendelig mange løsn.
1p.

<u>Konkl</u>: $A^2 \cdot \underline{x} = \underline{b}$ uendelig mange løsn. for $\underline{a = \pm 1}$.

<u>4.</u> Kostnadsfunksjon: $C(x)$
Enhetskostnad: $A(x) = \dfrac{C(x)}{x}$

$$\boxed{\text{Enhetskostnad minimal} \Rightarrow A'(x) = 0}$$

og $A'(x) = \left(\dfrac{C(x)}{x}\right)' = \dfrac{C'(x)\cdot x - C(x)\cdot 1}{x^2}$

$$= \frac{C'(x) - C(x)/x}{x} = \frac{C'(x) - A(x)}{x}$$

3p

<u>Konkl</u>: $A'(x) = 0 \Leftrightarrow \underline{C'(x) = A(x)}$

$C(x) = x^3 + 1200x + 16.000$
$C'(x) = 3x^2 + 1200$
$A(x) = \dfrac{C(x)}{x} = x^2 + 1200 + \dfrac{16.000}{x}$

$C' = A$: $3x^2 + \cancel{1200} = x^2 + \cancel{1200} + \dfrac{16.000}{x}$

$2x^2 = \dfrac{16000}{x} \quad | \cdot x$

$\dfrac{2x^3}{2} = \dfrac{16.000}{2}$

$x^3 = 8.000$

$x = \sqrt[3]{8000} = \underline{20}$

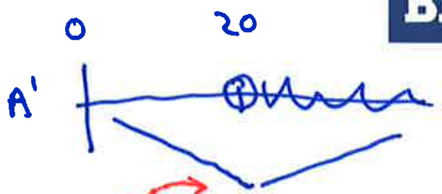
<u>Minimal enhetskostnad:</u>

$A(20) = 20^2 + 1200 + \dfrac{16.000}{20} = 400 + 1200 + 800$

$= \underline{\underline{2400}}$ 3p.

$$A'(x) = \frac{C'(x) - A(x)}{x} = \frac{(3x^2 + 1200) - (x^2 + 1200 + \frac{16000}{x})}{x}$$

$$= \frac{2x^2 - \frac{16000}{x}}{x} \quad \frac{\cdot x}{\cdot x} = \frac{2x^3 - 16000}{x^2}$$



Bør egentlig sjekke at $A'(x)=0$ gir minimum

5. $f(x,y) = xy^2 + x^3y - xy$

(a) $f'_x = y^2 + 3x^2y - y$ $\boxed{3p.}$

$f'_y = 2xy + x^3 - x$

$f'_x = 0: \quad y(y + 3x^2 - 1) = 0$

$f'_y = 0: \quad x(2y + x^2 - 1) = 0$

* $y=0, x=0 \to (0,0)$
* $y=0,$ $2y + x^2 - 1 = 0 \to x^2 = 1 \to x = \pm 1$ $(\pm 1, 0)$
* $y + 3x^2 - 1 = 0,$ $x = 0 \to y - 1 = 0 \to y = 1$ $(0,1)$
* $y + 3x^2 - 1 = 0,$ $2y + x^2 - 1 = 0 \to x^2 = 1 - 2y$
  $y + 3(1-2y) - 1 = 2 - 5y = 0$
  $y = 2/5$
  $\to x^2 = 1 - 2\cdot(2/5) = 1/5$
  $x = \pm\sqrt{1/5}$
  $(\pm\sqrt{1/5}, 2/5)$

Stasjonære pkt: $\boxed{3p.}$

$(0,0), (\pm 1, 0), (0,1),$
$(\pm\sqrt{1/5}, 2/5)$

(b) $f''_{xx} = 6xy \qquad f''_{xy} = 2y + 3x^2 - 1$

$f''_{yy} = 2x$

$\Downarrow$

$$H(f) = \begin{pmatrix} 6xy & 2y+3x^2-1 \\ 2y+3x^2-1 & 2x \end{pmatrix}$$ $\boxed{3p.}$

$\underline{(0,0)}$: $H(f)(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ $\det H = -1 < 0$ $\underline{\text{sadelpkt}}$

$(\pm 1, 0)$: $H(f)(\pm 1, 0) = \begin{pmatrix} 0 & 2 \\ 2 & \pm 2 \end{pmatrix}$ $\det H = -4 < 0$ $\underline{\text{sadelpkt}}$

$(0,1)$: $H(f)(0,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\det H = -1 < 0$ $\underline{\text{sadelpkt}}$

$(\pm\sqrt{1/5}, 2/5)$: $H(f)(\sqrt{1/5}, 2/5) = \begin{pmatrix} \pm 6\sqrt{1/5} \cdot 2/5 & 4/5 + 3/5 - 1 \\ 4/5 + 3/5 - 1 & \pm 2\sqrt{1/5} \end{pmatrix}$

$$= \begin{pmatrix} \pm \frac{12}{5}\sqrt{1/5} & 2/5 \\ 2/5 & \pm 2 \cdot \sqrt{1/5} \end{pmatrix}$$ $\det H = \frac{24}{5} \cdot \frac{1}{5} - \frac{4}{25} = \frac{20}{25} > 0$

$\Downarrow$

$(\sqrt{1/5}, 2/5)$ $\underline{\text{lokalt min}}$ $(A > 0)$

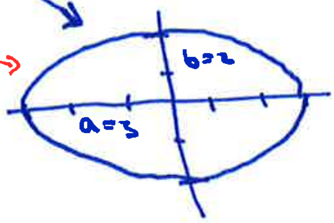
$(-\sqrt{1/5}, 2/5)$ $\underline{\text{lokalt maks}}$ $(A < 0)$

$\boxed{3p.}$

$\underline{6.}$ max $f(x,y) = 2x + 3y$ når $4x^2 + 9y^2 = 36$

(a) $4x^2 + 9y^2 = 36 \quad \Big| \cdot \frac{1}{36}$

$\boxed{3p.}$



$\frac{x^2}{9} + \frac{y^2}{4} = 1$ ellipse me senter $(0,0)$
av halvakser $a = \sqrt{9} = 3$
og $b = \sqrt{4} = 2$

Ellipsen er $\underline{\text{begrenset}}$ siden $\begin{cases} -3 \le x \le 3 \\ -2 \le y \le 2 \end{cases}$ $\boxed{3p.}$

(b) $L = 2x+3y - \lambda(4x^2+9y^2)$

Lagrange-betingelsene:

$L'_x = 0$: $2 - \lambda \cdot 8x = 0$ ← 2p.

$L'_y = 0$: $3 - \lambda \cdot 18y = 0$

$g(x,y) = a$: $4x^2+9y^2 = 36$

$2 = 8\lambda x$ $\qquad$ $3 = 18\lambda y$

$x = \frac{2}{8\lambda} = \underline{\frac{1}{4\lambda}}$ $\qquad$ $y = \frac{3}{18\lambda} = \underline{\frac{1}{6\lambda}}$

$4x^2+9y^2 = 4\cdot\left(\frac{1}{4\lambda}\right)^2 + 9\cdot\left(\frac{1}{6\lambda}\right)^2 = 36$

$\frac{4}{16\lambda^2} + \frac{9}{36\lambda^2} = 36 \quad | \cdot 4$

$\frac{1}{\lambda^2} + \frac{1}{\lambda^2} = 4\cdot 36 = 144$

$\frac{2}{\lambda^2} = 144 \qquad \lambda^2 = \frac{2}{144} = \frac{1}{72} \qquad \lambda = \pm\sqrt{\frac{1}{72}}$

$= \pm\sqrt{\frac{1}{2}\cdot\frac{1}{36}}$

$= \underline{\pm\frac{1}{6}\cdot\sqrt{\frac{1}{2}}}$

$\lambda = \frac{1}{6}\sqrt{\frac{1}{2}}$: $\quad x = \frac{1}{4\lambda} = \frac{1}{4}\cdot\sqrt{\frac{72}{1}} = \frac{6\cdot\sqrt{2}}{4} = \frac{3}{2}\sqrt{2}$

$y = \frac{1}{6\lambda} = \frac{1}{6}\cdot\sqrt{\frac{72}{1}} = \frac{6\cdot\sqrt{2}}{6} = \sqrt{2}$

$\Downarrow$

$\underline{(x,y;\lambda) = \left(\frac{3}{2}\sqrt{2}, \sqrt{2}; \frac{1}{6}\sqrt{\frac{1}{2}}\right)}$ $\qquad f = \underline{6\sqrt{2}}$ $\qquad$ 2p.

$\lambda = -\frac{1}{6}\sqrt{\frac{1}{2}}$: $\quad x = \frac{1}{4}\left(-\sqrt{\frac{72}{1}}\right) = -\frac{3}{2}\sqrt{2}$

$y = \frac{1}{6}\left(-\sqrt{\frac{72}{1}}\right) = -\sqrt{2}$

$\Downarrow$

$\underline{(x,y;\lambda) = \left(-\frac{3}{2}\sqrt{2}, -\sqrt{2}; -\frac{1}{6}\sqrt{1/2}\right)}$ $\qquad f = -6\sqrt{2}$

2p.

(c) $4x^2+9y^2=36$ er lukket (ulikhet) og begrenset (tra al), og derfor har problemet et maksimum fra ekstremverdisetningen. 2P.

Kandidater:

* Lagrange-betingelsene: $\boxed{\left(\frac{3}{2}\sqrt{2}, \sqrt{2}; \frac{1}{6}\sqrt{1/2}\right) \quad f = 6\sqrt{2}}$ ← beste kandidat for maks. blant disse punktene.

$\left(-\frac{3}{2}\sqrt{2}, -\sqrt{2}; -\frac{1}{6}\sqrt{1/2}\right) \quad f = -6\sqrt{2}$ 2P

* Degenerert bibetingelse: $4x^2+9y^2=36$ bibet.

$J = (8x \quad 18y) \Rightarrow$ $8x = 18y = 0$ degenerert bibetingelse

$\Downarrow$

$x = y = 0$ $(0,0)$

Dette pkt. er ikke tillatt siden

$4x^2+9y^2 = 0 \neq 36$

ingen kandidat med degenerert bibetingelse 2P

Konklusjon: $f_{maks} = 6\sqrt{2}$ i $(x,y) = \left(\frac{3}{2}\sqrt{2}, \sqrt{2}\right)$