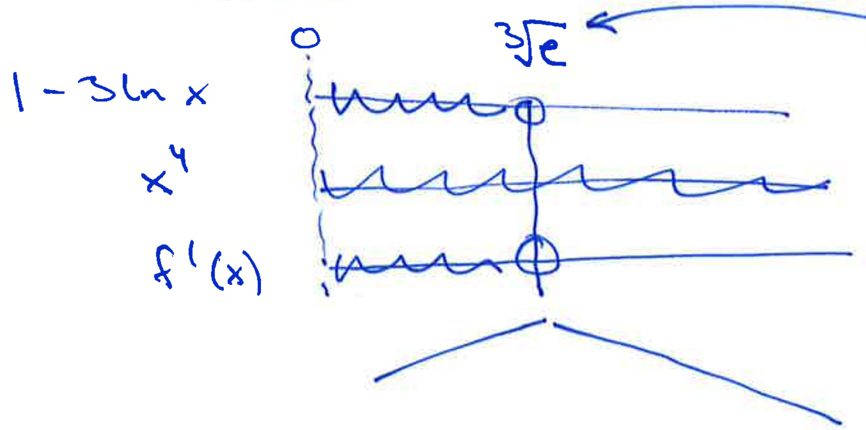
Løsning: Eksamenssett I

1. $f(x) = \dfrac{\ln x}{x^3}$, $x > 0$

(a) $f'(x) = \dfrac{1/x \cdot x^3 - \ln x \cdot 3x^2}{x^6} = \dfrac{x^2 - 3x^2 \ln x}{x^6} = \dfrac{\cancel{x^2}(1 - 3\ln x)}{\cancel{x^6}\, x^4}$

$= \dfrac{1 - 3\ln x}{x^4}$ $\boxed{3p.}$



$1 - 3\ln x = 0$
$\ln x = 1/3$
$x = e^{1/3}$
$= \sqrt[3]{e}$
$\simeq 1{,}40$

f voksende: $(0, \sqrt[3]{e}]$ $\boxed{3p.}$ (med fortegnsskjema)
f avtagende: $[\sqrt[3]{e}, \rightarrow)$

(b) Bruker $f'(x)$ og fortegnsskjema fra (a). Ser at:

$x = \sqrt[3]{e}$ lokalt og globalt maks.pkt. $\boxed{2p.}$
Maks. verdi: $f_{maks} = f(\sqrt[3]{e}) = \dfrac{\ln(e^{1/3})}{(e^{1/3})^3} = \dfrac{1/3}{e}$

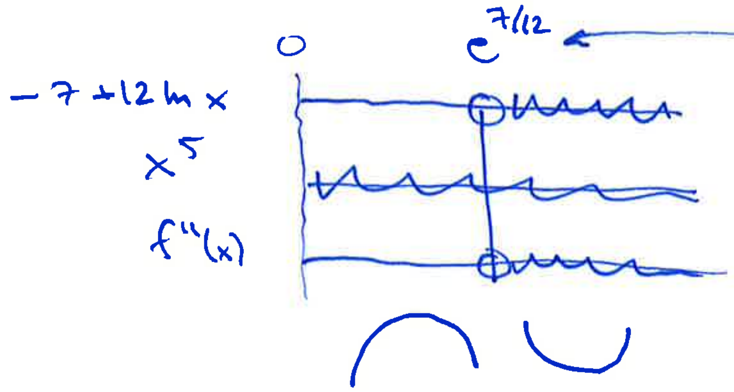
$= \dfrac{1}{3e} \simeq 0{,}12$

$\boxed{2p.}$

Ingen andre stasjonære pkt, randpkt, etc $\Rightarrow$
Ingen globale minimum $\boxed{2p.}$

(c) $f''(x) = \left(\dfrac{1-3\ln x}{x^4}\right)' = \dfrac{-3\cdot\frac{1}{x}\cdot x^4 - (1-3\ln x)\cdot 4x^3}{x^8}$

$= \dfrac{-3x^3 - 4x^3(1-3\ln x)}{x^8} = \dfrac{\cancel{x^3}(-3-4+12\ln x)}{\cancel{x^8}\, x^5} = \underline{\underline{\dfrac{-7+12\ln x}{x^5}}}$ $\boxed{3p.}$



$-7+12\ln x = 0$

$\ln x = 7/12$

$\underline{x = e^{7/12}}$

$\approx 1{,}79$

Vendepkt: $x = \underline{\underline{e^{7/12}}}$ $\boxed{2p.}$ (med fortegnsskjema)

f konveks i $[e^{7/12}, \to)$

f konkav i $(0, e^{7/12}]$ $\boxed{1p}$

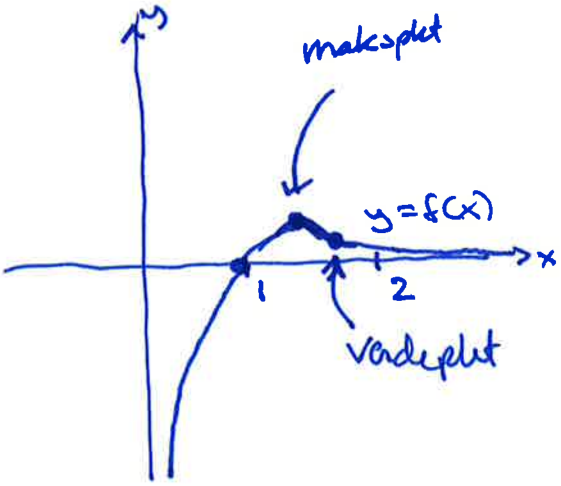
(d) $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{\ln x}{x^3} = \underline{\underline{-\infty}}$

$\ln x \to -\infty$
$x^3 \to 0^+$ $\boxed{2p.}$

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \underset{\text{"}\frac{\infty}{\infty}\text{"}}{\dfrac{\ln x}{x^3}} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}\cdot x}{3x^2\cdot x} = \lim\limits_{x\to\infty} \dfrac{1}{3x^3} = \underline{\underline{0}}$

$1 = 1$
$3x^3 \to \infty$

$\boxed{2p.}$



$\boxed{2p.}$ med maksepkt $x \simeq 1{,}40$, vendepkt $x \simeq 1{,}80$, grenseverdier

riktig skissert (evt nullpkt i x=1), resten trenger ikke være nøyaktig

2.

a) $\int (x+1)^3 dx = \int u^3 du = \frac{1}{4}u^4 + C$

$\boxed{\begin{aligned} u &= x+1 \\ du &= 1 \cdot dx \end{aligned}}$ 3p

$= \underline{\underline{\frac{1}{4}(x+1)^4 + C}}$ 3p.

b) $\int \overset{u'}{\overset{\shortparallel}{x}} \cdot \overset{v}{\overset{\shortparallel}{\ln x}}\, dx = u \cdot v - \int u \cdot v' \, dx$

$\boxed{\begin{aligned} u &= x^2/2 \quad & v &= \ln x \\ u' &= x & v' &= 1/x \end{aligned}}$ 3p.

$= \frac{1}{2}x^2 \cdot \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x} dx$

$= \frac{1}{2}x^2 \cdot \ln x - \frac{1}{2}\int x \, dx = \underline{\underline{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}}$ 3p.

c) $\int_1^5 \sqrt{2x-1}\, dx = \int_1^9 \sqrt{u} \cdot \frac{du}{2}$

$\boxed{\begin{aligned} u &= 2x-1 \\ du &= 2 \cdot dx \end{aligned}}$ 3p.

$u(1) = 2 \cdot 1 - 1 = 1$

$u(5) = 2 \cdot 5 - 1 = 9$

$= \frac{1}{2}\int_1^9 \sqrt{u}\, du = \left[\frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}u^{3/2}\right) + C\right]_1^9$ ← 2p.

$= \frac{1}{3} \cdot 9^{3/2} - \frac{1}{3} \cdot 1^{3/2}$

$= \frac{1}{3} \cdot 9 \cdot \sqrt{9} - \frac{1}{3} \cdot 1 \cdot \sqrt{1} = 9 - \frac{1}{3} = \underline{\underline{\frac{26}{3}}}$

1p.

d) $f(x) = 3x \geq g(x) = x^3 - 2 \; ; \; R$

Skjæringspkt: $3x = x^3 - 2$

$0 = x^3 - 3x - 2$ ← prøver $x = \pm 1, \pm 2$

$0 = \underbrace{(x+1)(x-2)}_{x^2 - x - 2} \cdot (\ldots)$

$x = -1$: ok

$x = 2$: ok.

poly. division for å finne

$(x^3 - 3x - 2) : (x^2 - x - 2) = \underline{x + 1}$

$\underline{-(x^3 - x^2 - 2x)}$

$x^2 - x - 2$

$\underline{x^2 - x - 2}$

0

$0 = (x+1)^2(x-2)$

Skjæringspkt: $\underline{x=-1}$, $\underline{x=2}$

$$\text{Areal} = \int_{-1}^{2} f(x)-g(x)\,dx = \underline{\int_{-1}^{2} 3x-(x^3-2)\,dx} \quad \boxed{4p.}$$

med utregning av skj. pkt.

$$= \int_{-1}^{2} 3x - x^3 + 2\,dx = \left[3\cdot\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 + 2x\right]_{-1}^{2}$$

$$= \left(\tfrac{3}{2}\cdot 2^2 - \tfrac{1}{4}\cdot 2^4 + 2\cdot 2\right) - \left(\tfrac{3}{2}(-1)^2 - \tfrac{1}{4}(-1)^4 + 2\cdot(-1)\right)$$

$$= (6-4+4) - (3/2 - 1/4 - 2) = 6 - (-3/4) = 6 + 3/4$$

$$= \underline{\underline{\frac{27}{4}}} = \underline{\underline{6{,}75}} \quad \boxed{2p.}$$

$\underline{3.}$

$$x + 3y - 4z = 2$$
$$3x - y + az = 4$$
$$4x + 2y - z = a$$

(a) $\underline{\underline{A = \begin{pmatrix} 1 & 3 & -4 \\ 3 & -1 & a \\ 4 & 2 & -1 \end{pmatrix}}}$ $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\underline{\underline{\underline{b} = \begin{pmatrix} 2 \\ 4 \\ a \end{pmatrix}}}$ $\boxed{6p}$ gir $A\underline{x} = \underline{b}$ for systemet ovenfor.

(b)
$$\begin{vmatrix} 1 & 3 & -4 \\ 3 & -1 & a \\ 4 & 2 & -1 \end{vmatrix} = 1\cdot\begin{vmatrix} -1 & a \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 3 & a \\ 4 & -1 \end{vmatrix} + (-4)\cdot\begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} \quad \boxed{3p.}$$

med raden som kofaktor-utvikles vist.

$$= 1(1-2a) - 3(-3-4a) - 4(\underbrace{6+4}_{10})$$

$$= 1 - 2a + 9 + 12a - 40$$

$$= \underline{10a - 30} = \underline{10(a-3)} \quad \boxed{3p}$$

(c) $\underline{a=2}$: $(A \mid \underline{b}) = \left(\begin{array}{ccc|c} 1 & 3 & -4 & 2 \\ 3 & -1 & 2 & 4 \\ 4 & 2 & -1 & 2 \end{array}\right)$ -3, -4

$a=2$

$\to \left(\begin{array}{ccc|c} 1 & 3 & -4 & 2 \\ 0 & -10 & 14 & -2 \\ 0 & -10 & 15 & -6 \end{array}\right)$ -1 $\to \left(\begin{array}{ccc|c} 1 & 3 & -4 & 2 \\ 0 & -10 & 14 & -2 \\ 0 & 0 & 1 & -4 \end{array}\right)$ $\boxed{3p.}$ med pivot-pos. vist.

$$\begin{aligned} \underline{x} + 3y - 4z &= 2 \\ -10\underline{y} + 14z &= -2 \\ \underline{z} &= -4 \end{aligned} \implies \underline{z = -4}$$

$$-10y = -2 - 14\cdot(-4) = 54 \implies y = \frac{54}{-10} = \underline{-\frac{27}{5}}$$

$$x = 2 - 3(-27/5) + 4(-4) = \frac{81}{5} - 14 = \underline{\frac{11}{5}}$$

$$(x, y, z) = \underline{\underline{(11/5, -27/5, -4)}}$$ $\boxed{3p.}$

$$\approx \underline{\underline{(2.2, -5.4, -4)}}$$

(d) Vet at $|A| \neq 0 \iff$ systemet har én løsn. (konsistent) $\boxed{2p}$

$\underline{|A| = 0}$: $|A| = 10a - 30 = 10(a-3)$ fra (b)

$|A| = 0$: $10(a-3) = 0$

$\underline{a=3}$ $\leftarrow$ $\left\{ \begin{array}{l} \text{ingen løsn.} \\ \text{eller} \\ \text{uendelig mange løsn.} \end{array} \right.$

$\boxed{2p.}$ med

$\underline{\text{Setter } a=3}$: $\left(\begin{array}{ccc|c} 1 & 3 & -4 & 2 \\ 3 & -1 & 3 & 4 \\ 4 & 2 & -1 & 3 \end{array}\right)$ -3, -4 $\to \left(\begin{array}{ccc|c} 1 & 3 & -4 & 2 \\ 0 & -10 & 15 & -2 \\ 0 & -10 & 15 & -5 \end{array}\right)$ -1

$\to \left(\begin{array}{ccc|c} 1 & 3 & -4 & 2 \\ 0 & -10 & 15 & -2 \\ 0 & 0 & 0 & -3 \end{array}\right)$ ingen løsn. for $\underline{a=3}$

$\underline{\text{Konklusjon}}$: systemet inkonsistent $\iff$ $\underline{\underline{a=3}}$ $\boxed{2p.}$ med

4. $f(x,y) = 12 - x^2 + xy - y^2 + 6x - 6y$

(a) $f'_x = -2x + y + 6$

$f'_y = x - 2y - 6$ $\boxed{3p.}$

Stasjonære pkt: $f'_x = 0$, $f'_y = 0$ $\Rightarrow$ $-2x + y = -6$, $x - 2y = 6$

$\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 4 - 1 = 3 \neq 0$ $\longrightarrow$ $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \end{pmatrix}$

den inverse matrisen fines.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -6 \\ 6 \end{pmatrix}$$

$$= \underbrace{\frac{1}{3} \cdot \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}}_{\frac{1}{|A|} \cdot adj(A)} \cdot \begin{pmatrix} -6 \\ 6 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 12 - 6 \\ 6 - 12 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 6 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Stasjonære pkt: $(x,y) = (2,-2)$ $\boxed{3p.}$

(kan også løses uten matriser)

(b) $f''_{xx} = -2$ $\quad f''_{xy} = 1$

$f''_{yx} = +1$ $\quad f''_{yy} = -2$

$\Downarrow$

$H(f) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ $\boxed{3p.}$

Klassifikasjon av $(2,-2)$:

Ser på $H(f)(2,-2) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ $\leftarrow$ konstant matrise

$\det = AC - B^2 = (-2)\cdot(-2) - 1^2 = 3 > 0$

$A, C = -2 < 0$

Andrederivert-testen: $(x,y) = (2,-2)$ er lokalt maks for f $\boxed{3p.}$ med andrederivert-testen

(c) $f(3,3) = 12 - 3^2 + 3\cdot 3 - 3^2 + 6\cdot 3 - 6\cdot 3 = 3$

$\Rightarrow (x,y) = (3,3)$ ligger på $\underline{f(x,y) = 3}$ (nivåkurve). $\boxed{2p.}$

Tangent:

$y - 3 = a \cdot (x-3)$

$y - 3 = \frac{1}{3}(x-3)$

$y - 3 = \frac{1}{3}x - 1$

$y = \underline{\underline{\frac{1}{3}x + 2}}$ $\boxed{1p.}$

der $a = y'(3,3) = -\dfrac{f'_x(3,3)}{f'_y(3,3)}$

$= -\dfrac{3}{-9} = \underline{+1/3}$

bruker f'_x, f'_y fra (a). $\boxed{3p.}$

(d) Hvis (x,y) er globalt min, er det også lokalt min. $\boxed{3p}$

Fra (b) er det <u>ingen ~~lokale~~ lokale min</u> (siden $(2,-2)$ eneste stasjonære pkt er lokalt <u>maks</u>) $\boxed{3p.}$

$\Downarrow$

<u>Ingen globale min</u>

<u>5.</u> $f(x) = \ln x$

(a) $L(x) = f(1) + f'(1)\cdot(x-1)$ $\boxed{3p.}$

$= 0 + 1\cdot(x-1) = \underline{\underline{x-1}}$

$f(1) = \ln 1 = \underline{0}$

$f'(x) = 1/x$

$\Rightarrow f'(1) = 1/1 = \underline{1}$ $\boxed{3p.}$

lokal approksimasjon i $\underline{x=1}$ til f,

dvs $f(x) = \ln(x) \simeq L(x) = \underline{x-1}$

(b) $$P_4(x) = f(1) + f'(1)\cdot(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f''''(1)}{4!}(x-1)^4$$

$$= 0 + 1\cdot(x-1) + \frac{(-1)}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 - \frac{6}{24}(x-1)^4$$

$$= \underline{\underline{(x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4}}$$

$\boxed{2p.}$

$f(x) = \ln x$

$f'(x) = 1/x$

$f''(x) = -1/x^2$

$f'''(x) = -1\cdot(-2)/x^3 = 2/x^3$

$f''''(x) = 2\cdot(-3)/x^4 = -6/x^4$

$f(1) = \underline{0}$

$f'(1) = \underline{1}$

$f''(1) = -1/1^2 = \underline{-1}$

$f'''(1) = 2/1^3 = \underline{2}$

$f''''(1) = -6/1^4 = \underline{-6}$

$\boxed{2p.}$

$$\ln(2) = f(2) \simeq P_4(2) = 1 - \tfrac{1}{2}\cdot 1^2 + \tfrac{1}{3}\cdot 1^3 - \tfrac{1}{4}\cdot 1^4$$

$$= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} = \frac{24-12+8-6}{24} = \frac{14}{24}$$

$$= \underline{\underline{\frac{7}{12}}} \simeq 0.5833$$

$\boxed{2p.}$