Løsning: MET11806 2022/03

1. a) $\int_0^1 12x^2+3\sqrt{x}\,dx = \left[12\cdot\frac{1}{3}x^3+3\cdot\frac{2}{3}x^{3/2}\right]_0^1$

$= \left[4x^3+2x\sqrt{x}\right]_0^1 = (4+2)-0 = \underline{\underline{6}}$

b) $\int 9\sqrt{x}\ln x\,dx = \int 9x^{1/2}\ln x\,dx = 6x\sqrt{x}\ln x - \int 6x^{3/2}\cdot\frac{1}{x}\,dx$

$$\boxed{\begin{array}{ll} u = 9\cdot\frac{2}{3}x^{3/2} & v=\ln x \\ u' = 9x^{1/2} & v' = 1/x \end{array}}$$

$= 6x\sqrt{x}\ln x - 6\cdot\frac{2}{3}x^{3/2}+C = 6x\sqrt{x}\ln x - 4x\sqrt{x}+C$

$\Rightarrow \int_1^2 9\sqrt{x}\ln x\,dx = \left[6x\sqrt{x}\ln x - 4x\sqrt{x}\right]_1^2$

$=(12\sqrt{2}\ln 2 - 8\sqrt{2})-(0-4\cdot 1) = \underline{\underline{12\sqrt{2}\ln 2 - 8\sqrt{2}+4}}$

c) $\frac{6}{x^2-1} = \frac{A}{x-1}+\frac{B}{x+1} \Rightarrow 6 = A(x+1)+B(x-1) \Rightarrow A=3$

$= \underbrace{(A+B)}_{=0}x+\underbrace{(A-B)}_{=6}$ $\quad B=-3$

$\int \frac{6}{x^2-1}\,dx = 3\ln|x-1|-3\ln|x+1|+C = 3\ln\left|\frac{x-1}{x+1}\right|+C$

$\Rightarrow \int_3^\infty \frac{6}{x^2+1}\,dx = \left[3\ln\left|\frac{x-1}{x+1}\right|\right]_3^\infty = 0-(3\ln(2/4))$

$= -3\ln(1/2) = -3(-\ln 2) = \underline{\underline{3\ln 2}}$

siden $\ln\left|\frac{x-1}{x+1}\right| \to \ln 1 = 0$ når $x\to\infty$.

d) $\int \frac{1}{\sqrt{x}+1}\,dx = \int \frac{1}{u}\,2\sqrt{x}\,du = \int \frac{1}{u}\,2(u-1)\,du = \int \frac{2u-2}{u}\,du$

$$\boxed{\begin{array}{l} u=\sqrt{x}+1 \\ du = \frac{1}{2\sqrt{x}}\,dx \end{array}}$$

$= \int 2-\frac{2}{u}\,du = 2u-2\ln|u|+C$

$= 2(\sqrt{x}+1)-2\ln(\sqrt{x}+1)+C$

$= \int_0^\infty \frac{1}{\sqrt{x}+1}\,dx = \left[2(\sqrt{x}+1)-2\ln(\sqrt{x}+1)\right]_0^\infty = \lim_{b\to\infty}\left(2(\sqrt{b}+1)-2\ln(\sqrt{b}+1)\right)$

$-(2\cdot 1-2\ln 1) = \underline{\underline{\infty}}$ siden $\sqrt{b}+1-\ln(\sqrt{b}+1)\to\infty$ når $b\to\infty$.

2.

a) $\frac{x+9}{x^2-3x-10} = \frac{A}{(x+2)} + \frac{B}{(x-5)} \Rightarrow x+9 = A(x-5)+B(x+2)$

$= \underbrace{(A+B)}_{=1}x + \underbrace{(-5A+2B)}_{=9}$

$A+B=1$
$-5A+2B=9$
$\Rightarrow A = 1-B$
$-5(1-B)+2B=9$
$\frac{7B}{7}=\frac{9+5}{7}$

$B=2$
$A=-1$

$\int \frac{x+9}{x^2-3x-10}dx = \int \frac{-1}{x+2} + \frac{2}{x-5}\,dx$

$= -\ln|x+2| + 2\ln|x-5| + C$

b) $\int \frac{\ln(x-1)}{x^2}dx = \int x^{-2}\ln(x-1)\,dx = -x^{-1}\ln(x-1) - \int -x^{-1}\frac{1}{x-1}dx$

$u = -x^{-1}$ $\quad v = \ln(x-1)$
$u' = x^{-2}$ $\quad v' = \frac{1}{x-1}$

$= -\frac{1}{x}\ln(x-1) + \int \frac{1}{x(x-1)}dx$

$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$1 = A(x-1)+Bx$
$= \underbrace{(A+B)}_{=0}x + \underbrace{(-A)}_{=1}$

$A=-1,\ B=1$

$= -\frac{1}{x}\ln(x-1) + \int \frac{-1}{x} + \frac{1}{x-1}dx$

$= -\frac{1}{x}\ln(x-1) - \ln x + \ln(x-1) + C$

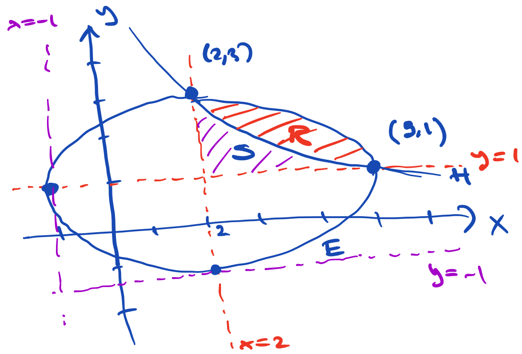
c) $\int \frac{1}{e^x+1}dx = \int \frac{1}{u}\cdot\frac{1}{e^x}du = \int \frac{1}{u}\cdot\frac{1}{(u-1)}du$

$u = e^x+1$
$du = e^x dx$

Fra (b) $= \int \frac{-1}{u} + \frac{1}{u-1}\,du = -\ln|u| + \ln|u-1|$

$= -\ln(e^x+1) + \ln(e^x) + C = x - \ln(e^x+1) + C$

3.



a) E: $\frac{(x-2)^2}{3^2} + \frac{(y-1)^2}{2^2} = 1$ eller $\underline{\underline{4(x-2)^2 + 9(y-1)^2 = 36}}$

H: $(x+1)(y+1) = c$ $\Rightarrow$ $(x-1)(y-1) = 12$

(2,3): $(2+1)(3+1) = c$ eller

$3 \cdot 4 = c$ $y+1 = \frac{12}{x+1}$

$c = 12$ $\underline{\underline{y = -1 + \frac{12}{x+1}}}$

b) Se figur ovenfor, med området R markert.
La oss kalle området begrenset av $x=2$, $y=1$ og H for S,
slik at $A(R) + A(S) = \frac{A(\text{ellipsen})}{4} = \frac{\pi \cdot 3 \cdot 2}{4} = \frac{6}{4}\pi = \frac{3}{2}\pi$

Vi har at
$$A(S) = \int_2^5 \left(-1 + \frac{12}{x+1}\right) - 1 \, dx = \int_2^5 \frac{12}{x+1} - 2 \, dx = \Big[12\ln(x+1) - 2x\Big]_2^5$$
$$= (12\ln 6 - 10) - (12\ln 3 - 4) = 12\ln\left(\frac{6}{3}\right) - 10 + 4 = \underline{12\ln 2 - 6}$$

Dermed har området R areal:
$$A(R) = \frac{3}{2}\pi - A(S) = \frac{3}{2}\pi - (12\ln 2 - 6) = \underline{\underline{\frac{3}{2}\pi - 12\ln 2 + 6}}$$
$$\simeq 2.39$$

4.

a) NV: $\int_0^{10} f(x) e^{-rx} \, dx = \int_0^{10} (100+4x) e^{-0.10x} \, dx$

$$\boxed{\begin{array}{ll} u = -10e^{-0.1x} & v = 100+4x \\ u' = e^{-0.10x} & v' = 4 \end{array}}$$

$$= \Big[-10e^{-0.1x} \cdot (100+4x)\Big]_0^{10} - \int_0^{10} -10e^{-0.1x} \cdot 4 \, dx$$

$$= \Big[-10(100+4x)e^{-0.1x} + 40(-10)e^{-0.1x}\Big]_0^{10}$$

$$= \left(-1400e^{-1} - 400e^{-1}\right) - \left(-1000e^{0} - 400e^{0}\right) = 1400 - \frac{1800}{e} \simeq \underline{\underline{737.8}}$$

b) **NV:** $\displaystyle\int_0^{10} f(x)e^{-rx}\,dx = \int_0^{10} 100\cdot 1{,}04^{x}\, e^{-0.10x}\,dx$

$$= \int_0^{10} 100\, e^{\ln(1{,}04)x - 0.10x}\,dx = \int_0^{10\ln(1{,}04)-1} 100\, e^{u}\cdot \frac{du}{\ln 1{,}04 - 0.1}$$

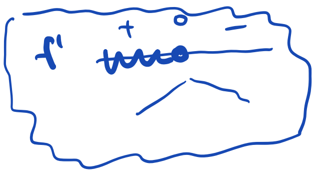
$$\boxed{\begin{array}{l} u = \ln(1{,}04)x - 0.10x \\ du = (\ln(1{,}04) - 0.10)\,dx \end{array}}$$

$$= \frac{100}{\ln 1{,}04 - 0{,}10}\Big[e^{u}\Big]_0^{10\ln(1{,}04)-1}$$

$$= \frac{100}{\ln 1{,}04 - 0{,}10}\left(e^{10\ln(1{,}04)-1} - 1\right)$$

$$= \underline{\underline{\frac{100}{\ln(1{,}04) - 0{,}10}\left(\frac{1{,}04^{10}}{e} - 1\right)}} \approx 749.3$$

5. Lokale maks der $f'(x) = 0$ og $f'(x)$ skifter fortegn fra + til ÷:



Ser fra figur at dette inntreffer for

$\underline{x=0}$ og $\underline{x=3}$

Vi har $f(3) > f(0)$ siden $f(3) - f(0) = \displaystyle\int_0^3 f'(x)\,dx$

$$= -A_0 + A_1 = A_1 - A_0 > 0$$

Vi ser fra figuren at $\underline{A_1 > A_0}$, eller $A_1 - A_0 > 0$. Dette betyr at $\underline{\underline{x=3}}$ er maks. punktet til f.

arealet over grafen til $f'(x)$ i $[0,1]$

arealet under grafen til $f'(x)$ i $[1,3]$

<u>Avlesning fra grafen:</u> $A_0 \approx 2$ ruter $= 2\cdot(1/4)^2 = 2/16 = 1/8 = \underline{0.125}$

$A_1 \approx 26$ ruter $= 26\,(1/4)^2 = 26/16 = 13/8 = \underline{1.625}$

Dermed er $\displaystyle\int_0^3 f'(x)\,dx \approx -0{,}125 + 1.625 = \underline{\underline{1.5}}$

<u>Merk:</u> Vi antar at f er definert for x i $[-1.5, 3.25]$, området vist på figuren. Da er randpkt. $x = -1.5$ også lokalt maks for f, men $\displaystyle\int_{-1.5}^3 f'(x)\,dx > 0$ så $f(-1.5) < f(3)$.

6.

a)

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 5 & 8 & -2 & 23 \\ 2 & 6 & -5 & 6 \\ 4 & 4 & 2 & 21 \end{array}\right)\begin{matrix} \\ -5 \\ -2 \\ -4 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & 8 \\ 0 & 2 & -3 & 0 \\ 0 & -4 & 6 & 9 \end{array}\right)\begin{matrix} \\ \\ 1 \\ -2 \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & 8 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & -7 \end{array}\right) \begin{matrix} \\ \\ \\ 7/8 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & 8 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

ingen løsning

trappeform

b)

$$\left(\begin{array}{cccc|c} 1 & 2 & 4 & 1 & 11 \\ 2 & 5 & 4 & -3 & 18 \\ 4 & 8 & 12 & 0 & 28 \end{array}\right)\begin{matrix} \\ -2 \\ -4 \end{matrix} \to \left(\begin{array}{cccc|c} 1 & 2 & 4 & 1 & 11 \\ 0 & 1 & -4 & -5 & -4 \\ 0 & 0 & -4 & -4 & -16 \end{array}\right)$$

trappeform

uendelig mange løsninger
(en frihetsgrad,
w fri)

$$\begin{aligned} x + 2y + 4z + w &= 11 \\ y - 4z - 5w &= -4 \\ -4z - 4w &= -16 \end{aligned}$$

Bakleng substitusjon:

$$\frac{-4z}{-4} = \frac{4w - 16}{-4}$$

$$z = -w + 4$$

$$\begin{aligned} y &= -4 + 4z + 5w \\ &= -4 + 4(-w+4) + 5w \\ &= w + 12 \end{aligned}$$

$$\begin{aligned} x &= 11 - 2y - 4z - w \\ &= 11 - 2(w+12) - 4(-w+4) - w \\ &= w - 29 \end{aligned}$$

$$\Rightarrow (x, y, z, w) = (w - 29,\ w + 12,\ -w + 4,\ w)$$ der w er fri

7. a) $\begin{vmatrix} 2 & 1 & 4 \\ 3 & 2 & 1 \\ 1 & a & 7 \end{vmatrix} = 2(14-a) - 1(21-1) + 4(3a-2)$

$= 28 - 2a - 20 + 12a - 8 = \underline{\underline{10a}}$

$|A| = 0$: $10a = 0$

$\underline{\underline{a = 0}}$

b) $\begin{vmatrix} 0 & s & 1 \\ s & 0 & 1 \\ 1 & 1 & s \end{vmatrix} = -s(s^2-1) + 1 \cdot s = -s^3 + 2s$

$= \underline{\underline{-s(s^2-2)}}$

$|A| = 0$: $-s(s^2-2) = 0$

$\underline{\underline{s = 0}}$ eller $s = \underline{\underline{\pm\sqrt{2}}}$

c) $\left|\begin{array}{cc|cc} 1 & t & 0 & 0 \\ t & 1 & 0 & 0 \\ \hline 0 & 0 & t & 8 \\ 0 & 0 & 2 & t \end{array}\right| = \begin{vmatrix} 1 & t \\ t & 1 \end{vmatrix} \cdot \begin{vmatrix} t & 8 \\ 2 & t \end{vmatrix} = \underline{\underline{(1-t^2) \cdot (t^2-16)}}$

$|A| = 0$: $(1-t^2) \cdot (t^2-16) = 0$

$t = \underline{\underline{\pm 1}}$ eller $t = \underline{\underline{\pm 4}}$

8. a) $\underline{w}$ lineær-komb. av $\underline{u}_1, \underline{u}_2, \underline{u}_3$ $\Longrightarrow$ $x\underline{u}_1 + y\underline{u}_2 + z\underline{u}_3 = \underline{w}$ har løsning

Lineært system:

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 3 & 5 & 11 & 4 \\ 2 & 6 & 4 & 6 \\ 4 & 7 & 9 & 2 \end{array}\right) \begin{matrix} \\ -3 \\ -2 \\ -4 \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & -1 & -4 & -5 \\ 0 & 2 & -6 & 0 \\ 0 & -1 & -11 & -10 \end{array}\right) \begin{matrix} \\ \\ 2 \\ -1 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & -1 & -4 & -5 \\ 0 & 0 & -14 & -10 \\ 0 & 0 & -7 & -5 \end{array}\right) \begin{matrix} \\ \\ \\ -1/2 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & -1 & -4 & -5 \\ 0 & 0 & -14 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\underline{\underline{\text{Ja}}}$, $\underline{w}$ er en lin. komb. av $\underline{u}_1, \underline{u}_2, \underline{u}_3$ siden systemet har en løsn.

b)

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & a \\ 3 & 5 & 11 & b \\ 2 & 6 & 4 & c \\ 4 & 7 & 9 & d \end{array}\right) \cdot(-3),\ \cdot(-2),\ \cdot(-4) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 5 & a \\ 0 & -1 & -4 & b-3a \\ 0 & 2 & -6 & c-2a \\ 0 & -1 & -11 & d-4a \end{array}\right) \cdot 2,\ \cdot(-1)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 5 & a \\ 0 & -1 & -4 & b-3a \\ 0 & 0 & -14 & c-2a+2(b-3a) \\ 0 & 0 & -7 & d-4a-1(b-3a) \end{array}\right) \cdot 2$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 5 & a \\ 0 & -1 & -4 & b-3a \\ 0 & 0 & -14 & c+2b-8a \\ 0 & 0 & -14 & 2d-2b-2a \end{array}\right) \cdot(-1)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 5 & a \\ 0 & -1 & -4 & b-3a \\ 0 & 0 & -14 & c+2b-8a \\ 0 & 0 & 0 & 2d-2b-2a-1\cdot(c+2b-8a) \end{array}\right)$$

Vi har én løsning $\Longleftrightarrow$ $2d - c - 4b + 6a = 0$ (ingen løsn. ellers)

$$\underline{c = 6a - 4b + 2d}$$

Alle lineærkomb. av $\underline{v}_1, \underline{v}_2, \underline{v}_3$: alle vektorer (a,b,c,d) slik at

$$\underline{\underline{c = 6a - 4b + 2d}}$$

9. a) $a=0$

$$A = \begin{pmatrix} 3 & 7 & 0 \\ 2 & 5 & 3 \\ 5 & 0 & 35 \end{pmatrix}$$

$$|A| = 3(5\cdot 35) - 7\cdot(70-15)$$
$$= 3\cdot 175 - 7\cdot 55 = 525 - 385 = \underline{140} \neq 0$$

$$A^{-1} = \frac{1}{140}\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{140}\begin{pmatrix} 175 & -55 & -25 \\ -245 & 105 & 35 \\ 21 & -9 & 1 \end{pmatrix}^T$$

$$= \underline{\underline{\frac{1}{140}\begin{pmatrix} 175 & -245 & 21 \\ -55 & 105 & -9 \\ -25 & 35 & 1 \end{pmatrix}}}$$

b) $|A| \neq 0 \iff A\underline{x} = \underline{b}$ har entydig løsning

$$\begin{vmatrix} 3 & 7 & a \\ 2 & 5 & 3 \\ 5 & a & 35 \end{vmatrix} = 3\cdot(175-3a) - 7(70-15) + a(2a-25)$$

$$= 2a^2 - 34a + 525 - 385$$
$$= 2(a^2 - 17a + 70) = \underline{2(a-7)(a-10)}$$

Entydig løsning: $\underline{\underline{a \neq 7, 10}}$

c) $\underline{a=7}$:

$$\left(\begin{array}{ccc|c} 3 & 7 & 7 & -8 \\ 2 & 5 & 3 & 4 \\ 5 & 7 & 35 & -144 \end{array}\right) \begin{matrix}]\cdot 1 \\ \\ \\ \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & -12 \\ 2 & 5 & 3 & 4 \\ 5 & 7 & 35 & -144 \end{array}\right) \begin{matrix}]\cdot 2 \\ ]\cdot 5 \\ \\ \end{matrix}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & -12 \\ 0 & 1 & -5 & 28 \\ 0 & -3 & 15 & -84 \end{array}\right)]3 \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & -12 \\ 0 & 1 & -5 & 28 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\underline{\text{Uendelig mange løsn.}}$ for $\underline{a=7}$

$$x + 2y + 4z = -12$$
$$y - 5z = 28$$

$$x = -12 - 4z - 2(5z+28) = \underline{-14z - 68}$$
$$y = \underline{5z + 28}$$

$$\underline{\underline{(x,y,z) = (-14z-68, 5z+28, z)}}$$ der z er fri

$\underline{a=10}$:

$$\left(\begin{array}{ccc|c} 3 & 7 & 10 & -8 \\ 2 & 5 & 3 & 4 \\ 5 & 10 & 35 & -144 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 7 & -12 \\ 2 & 5 & 3 & 4 \\ 5 & 10 & 35 & -144 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 7 & -12 \\ 0 & 1 & -11 & 28 \\ 0 & 0 & 0 & -84 \end{array}\right)$$

ingen løsn. for $a=10$

d) Entydig løsn: $a \neq 7, 10$

$$z = \frac{\begin{vmatrix} 3 & 7 & -8 \\ 2 & 5 & 4 \\ 5 & a & -144 \end{vmatrix}}{|A|} = \frac{-28a+196}{2(a-7)(a-10)} = \frac{-28\cancel{(a-7)}}{2\cancel{(a-7)}(a-10)} = \frac{-14}{a-10}$$

$$\begin{vmatrix} 3 & 7 & -8 \\ 2 & 5 & 4 \\ 5 & a & -144 \end{vmatrix} = 3(-720-4a) - 7(-288-20) - 8(2a-25)$$

$$= -28a+196$$

10. Vi skriver: $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$D = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$

$|A| = -1-1 = -2 \neq 0$

$A^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}A$

Matriselikning:

$AXA = D \quad | A^{-1}\cdot$

$A^{-1}AXA = A^{-1}D$

$XA = A^{-1}D \quad | \cdot A^{-1}$

$XAA^{-1} = A^{-1}DA^{-1}$

$X = A^{-1}DA^{-1} = \frac{1}{2}A \cdot D \cdot \frac{1}{2}A = \frac{1}{4}ADA$

$= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} 6 & 2 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}}$