Plan

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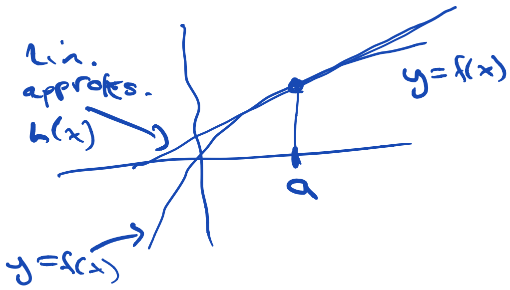
① Lineær approksimasjon

En variabel: $y = f(x)$, $x = a$

$$L(x) = f(a) + f'(a) \cdot (x - a)$$



To variabler: $z = f(x,y)$, (x_0, y_0)

$$\boxed{L(x,y) = f(x_0,y_0) + f'_x(x_0,y_0) \cdot (x - x_0) + f'_y(x_0,y_0) \cdot (y - y_0)}$$

Eks: $f(x,y) = \ln(x^2+y^2+1)$, i pkt. $(1,1)$

$$f'_x = \frac{1}{u} \cdot u'_x = \frac{2x}{x^2+y^2+1}$$

$$f'_y = \frac{1}{u} \cdot u'_y = \frac{2y}{x^2+y^2+1}$$

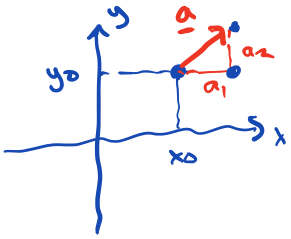
$$L(x,y) = f(1,1) + f'_x(1,1) \cdot (x-1) + f'_y(1,1) \cdot (y-1)$$

$$= \ln(3) + \frac{2}{3}(x-1) + \frac{2}{3}(y-1)$$

← lineær approx til $f(x,y)$ i $(1,1)$

2 Retningsderiverte

$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ retning



Defn: $f'_{\underline{a}} = a_1 \cdot f'_x + a_2 \cdot f'_y$

$f'_{a}(x_0, y_0) = a_1 \cdot f'_x(x_0, y_0) + a_2 \cdot f'_y(x_0, y_0)$

$\boxed{f'_{\underline{a}} = \underline{a} \cdot \nabla f} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} f'_x \\ f'_y \end{pmatrix}$

$= a_1 \cdot f'_x + a_2 \cdot f'_y$

$f'_x(x_0, y_0)$: endring i x-retn

$f'_y(x_0, y_0)$: — " — y-retn

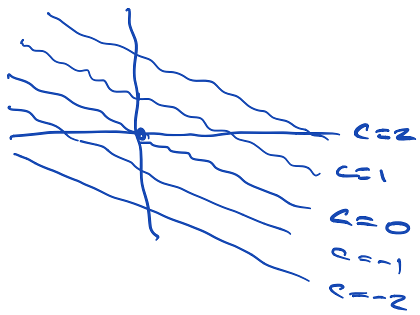
Oppgavesett 26

3 a) $f(x,y) = 2x + 3y \qquad c = -2, -1, 0, 1, 2$

$f(x,y) = c: \quad 2x + 3y = c$

$3y = c - 2x$

$\underline{y = c/3 - 2/3 \cdot x}$

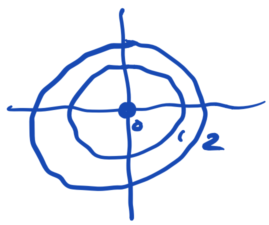


b) $f(x,y) = x^2 + y^2 = c$

sirkel m/r $= \sqrt{c}$ når $c > 0$

pkt $(0,0)$ når $c = 0$

ingen pkt $c < 0$



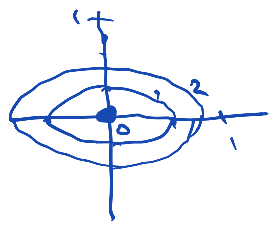
c) $f(x,y) = 4x^2 + 9y^2 = c \qquad 1 : c$

$\frac{4x^2}{c} + \frac{9y^2}{c} = 1$

$\frac{x^2}{c/4} + \frac{y^2}{c/9} = 1$

ellipse når $c > 0$

$a = \sqrt{c}/2, \; b = \sqrt{c}/3$



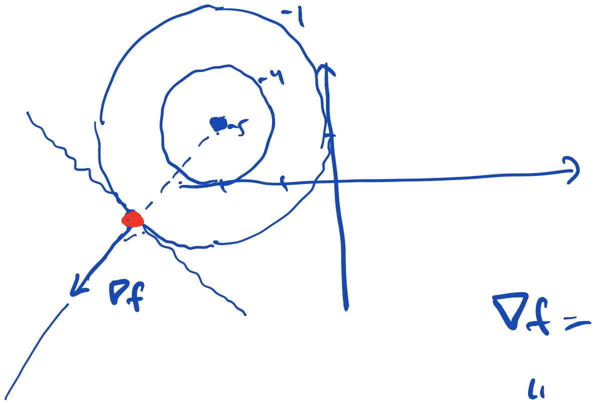
4. $f(x,y) = x^2+4x+y^2-2y = c \quad | +4+1$

$(x+2)^2 + (y-1)^2 = c+5$

$c > -5$: $\boxed{\text{sirkel sentrum } (-2,1) \text{ og } r=\sqrt{c+5}}$

$c = -5$: pkt $(-2,1)$

$c < -5$: ingen pkt



$\nabla f = \begin{pmatrix} 2x+4 \\ 2y-2 \end{pmatrix}$ er en vektor som peker radielt ut fra sentrum $(-2,1)$

$\parallel$

$\begin{pmatrix} f'_x \\ f'_y \end{pmatrix}$

∇f står normalt på (90°) førsteken i punktet

∇f peker i den retningen der f vokser raskest.

Eksamen 12/2017, Oppg 4

$f(x,y) = x^2y^2 + xy + x - y$

a) $f'_x = 2xy^2 + y + 1$

$f'_y = 2x^2y + x - 1$

$H(f) = \begin{pmatrix} 2y^2 & 4xy+1 \\ 4xy+1 & 2x^2 \end{pmatrix}$

$(x,y)=(-1,1)$: $\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$

b) N: $f(x,y) = 2$

L: $y = x$

$\boxed{\begin{aligned} x^2y^2+xy+x-y &= 2 \\ y &= x \end{aligned}}$ $y = x$

$x^2x^2 + x^2 = 2$

$x^4 + x^2 - 2 = 0$

$u = x^2$: $u^2 + u - 2 = 0$

$(u+2)(u-1) = 0$

$u = -2$ eller $u = 1$

$\cancel{x^2 = -2}$ eller $x^2 = 1$ $\rightarrow x = \pm 1$

To pkt: $(x,y) = (1,1), (-1,-1)$

c) Tangenter til N i (1,1), (−1,−1):

(1,1): $y-1 = y'(1,1)\cdot(x-1) \Rightarrow y-1=-2(x-1) \Rightarrow y = \underline{-2x+3}$

(−1,−1): $y+1 = y'(-1,-1)\cdot(x+1) \Rightarrow y+1 = -\frac{1}{2}(x+1) \Rightarrow y = \underline{-\frac{1}{2}x - \frac{3}{2}}$

$$\boxed{y-y_0 = a\cdot(x-x_0)}$$

$f(x,y)=2$: $x^2y^2+xy+x-y=2$

$$y' = -\frac{f'_x}{f'_y} = -\frac{2xy^2+y+1}{2x^2y+x-1}$$

$$y'(1,1) = -\frac{4}{2} = \underline{-2}$$

$$y'(-1,-1) = -\frac{-2+0}{-2-2} = \underline{-\frac{1}{2}}$$

d) $f'_x = 2xy^2+y+1=0$

$f'_y = 2x^2y+x-1=0$

$2xy^2+2x^2y+y+x+1-1=0+0$

$2xy(y+x)+(y+x)=0$

$(2xy+1)(y+x)=0$

$\Rightarrow$ $\underline{y=-x}$ eller $\underline{2xy=-1}$

$2x^3-x+1=0$

$\underline{x=-1}$ ok

$(2x^3-x+1):(x+1) = 2x^2-2x+1$

$-(2x^3+2x^2)$

$-2x^2-x+1$

$-(-2x^2-2x)$

$x+1$

$x+1$

0

$(x+1)(2x^2-2x+1)=0$

$\boxed{x=-1,\ y=1}$ eller $x = \cancel{\frac{2\pm\sqrt{4-4\cdot 2\cdot 1}}{2\cdot 2}}$

$2xy=-1$: $x = \frac{-1}{2y}$

$2xy^2+y+1=0$

$\cancel{2}\left(-\frac{1}{\cancel{2}y}\right)y^{\cancel{2}}+y+1=0$

$-y+y+1=0$

$1=0$

umulig.

Stasjonære pkt: $(x,y)=(-1,1)$

$$H(f)(-1,1) = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} \qquad \det = 4-9=-5<0$$

$(x,y)=(-1,1)$ er **sadelpkt**

d) $\underline{\text{Alt:}}$ $\underline{\text{Stasjonære pkt}}$

$$2xy^2 + y + 1 = 0$$
$$2x^2y + x - 1 = 0$$

$$\frac{2xy^2}{2y^2} = \frac{-y-1}{2y^2} \qquad x = \frac{-y-1}{2y^2} = -\frac{(y+1)}{2y^2}$$

$$2x^2y + x - 1 = 0$$

$$2 \cdot \frac{(y+1)^2}{4y^4} \cdot y - \frac{(y+1)}{2y^2} - 1 = 0 \qquad | \cdot 2y^3$$

$$(y+1)^2 - y(y+1) - 2y^3 = 0$$

$$\cancel{y^2} + 2y + 1 - \cancel{y^2} - y - 2y^3 = 0$$

$$-2y^3 + y + 1 = 0 \quad \leftarrow y = 1 \quad \Rightarrow (x,y) = (-1,1)$$

Eksamen 08/2012, Opps 4.

$$f(x,y) = \frac{2x+3y-6}{xy} = \frac{2x}{xy} + \frac{3y}{xy} - \frac{6}{xy} = \frac{2}{y} + \frac{3}{x} - \frac{6}{xy} = 2 \cdot y^{-1} + 3x^{-1} - 6 \cdot x^{-1}y^{-1}$$

a) $f'_x = 3 \cdot (-1)x^{-2} - 6y^{-1} \cdot (-1)x^{-2}$

$$= -\frac{3 \cdot y}{x^2 \cdot y} + \frac{6}{x^2y} = \frac{6-3y}{x^2y} = 0$$

$f'_y = 2 \cdot (-1)y^{-2} - 6x^{-1} \cdot (-1)y^{-2}$

$$= -\frac{2 \cdot x}{y^2 \cdot x} + \frac{6}{xy^2} = \frac{6-2x}{xy^2} = 0$$

$6 - 3y = 0 \quad y = 2$

$6 - 2x = 0 \quad x = 3$

$\Downarrow$

Stasjonære pkt: $(x,y) = (3,2)$

b) $f''_{xx} = -3(-2)x^{-3} + 6y^{-1}(-2)x^{-3}$

$$= \frac{6 \cdot y}{x^3 \cdot y} - \frac{12}{x^3y} = \frac{6y-12}{x^3y}$$

$$f''_{xy} = 6x^{-2} \cdot (-1)y^{-2} = -\frac{6}{x^2y^2}$$

$f''_{yy} = -2 \cdot (-2)y^{-3} + 6x^{-1}(-2)y^{-3}$

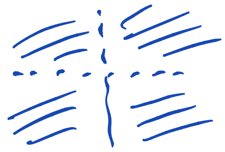
$$= \frac{4 \cdot x}{y^3 \cdot x} - \frac{12}{xy^3} = \frac{4x-12}{xy^3}$$

$$H(f)(3,2) = \begin{pmatrix} 0 & -1/6 \\ -1/6 & 0 \end{pmatrix}$$

$\det = 0 - 1/36 < 0$

$(3,2)$ er sadelpkt

c) max / min $f(x,y)$, D_f: $xy \neq 0$



Randpkt = aksene er ikke med i D_f.

Kandidatpkt: $(x,y) = (3,2)$ sadelpkt

ingen andre kandidatpkt

$\Downarrow$

ingen maks / ingen min.

d) N: $f(x,y) = 5$

L: $y = 1$

$$\frac{2x+3y-6}{xy} = 5$$
$$y = 1$$
$$\Downarrow$$
$$\frac{2x+3\cdot 1-6}{x\cdot 1} = 5 \quad | \cdot x$$
$$2x - 3 = 5x$$
$$-3x = 3$$
$$x = -1$$

NM: $(x,y) = (-1,1)$

$$y - 1 = y'(-1,1)\cdot(x+1)$$
$$y - 1 = 3/8(x+1)$$
$$y = \frac{3}{8}x + \frac{3}{8} + 1$$
$$y = \frac{3}{8}x + \frac{11}{8}$$

$$y' = -\frac{f'_x}{f'_y} = -\frac{\frac{6-3y}{x^2y}\cdot x^2y^2}{\frac{6-2x}{xy^2}\cdot x^2y^2} = -\frac{(6-3y)y}{(6-2x)\cdot x}$$

$$y'(-1,1) = -\frac{3\cdot 1}{8\cdot(-1)} = 3/8$$

Oppgavesett 28

5b) $f(x,y) = x^2y + xy^3 + xy^2$

$$\boxed{\begin{aligned} f'_x &= 2xy + y^3 + y^2 = 0 \\ f'_y &= x^2 + 3xy^2 + 2xy = 0 \end{aligned}}$$

Forelesning 21, Del 2

Stasjonære pkt:

$(x,y) = (0,0), (0,-1),$

$(3/25, -3/5)$

$$H(f) = \begin{pmatrix} 2y & 2x + 3y^2 + 2y \\ * & 6xy + 2x \end{pmatrix}$$

det = 0 kan ikke bruke andrederivert-testen

$(0,0)$: $H(f)(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$(0,-1)$: $H(f)(0,-1) = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$ det $= 0 - 1$ $= -1 < 0$ Sadelpkt

$(3/25, -3/5)$: $H(f)(3/25, -3/5) = \begin{pmatrix} -6/5 & 3/25 \\ 3/25 & -24/125 \end{pmatrix}$

$\det = -\frac{6}{5} \cdot \frac{-24}{125} - \left(\frac{3}{25}\right)^2$

$= \frac{144 - 9}{25^2} > 0$

$\mathrm{tr} = a + c < 0$

$\Downarrow$

lokalt maks

$2 \cdot 3/25 + 3 \cdot (-3/5)^2 + 2 \cdot (-3/5)$

$= \frac{6}{25} + \frac{27}{25} + \frac{-6 \cdot 5}{5 \cdot 5} = 3/25$

$2x\,(3y+1) = \frac{6}{25} \cdot \left(-\frac{9}{5} + \frac{5}{5}\right) = \frac{-24}{125}$

$(0,0)$: $f(0,0) = 0$

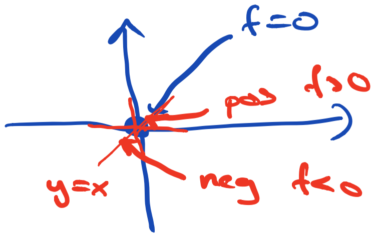
$f(x,y) = x^2y + xy^3 + xy^2$

$f(a,a) = a^3 + a^4 + a^3 = 2a^3 + a^4$

$= a^3(2 + a)$

pos $a>0$ pos.

neg $a<0$



(a,a)

$(0,0)$ er sadelpkt

c) $f(x,y) = \sqrt{36 - 9x^2 - 4y^2} = \sqrt{u}$, $u = 36 - 9x^2 - 4y^2$

$f'_x = \frac{1}{2\sqrt{u}} \cdot u'_x = \frac{-18x}{2\sqrt{u}} = \frac{-9x}{\sqrt{u}} = 0$

$f'_y = \frac{1}{2\sqrt{u}} \cdot u'_y = \frac{-8y}{2\sqrt{u}} = \frac{-4y}{\sqrt{u}} = 0$

$-9x = 0 \Rightarrow x = 0$

$-4y = 0 \quad y = 0$

Stasjonære pkt:

$(x,y) = (0,0)$

$f(0,0) = \sqrt{36} = \underline{6}$

$f(x,y) = \sqrt{36 - 7x^2 - 4y^2} \leq \sqrt{36} = 6$ for alle (x,y).

$\Downarrow$

$(0,0)$ er globalt maks for f

$\Downarrow$

$\underline{(0,0) \text{ er lokalt maks for } f}$

Alt: Bruke $H(f)(0,0)$ og andrederivert-testen.