Plan

1. Tolkning av Lagrange-multiplikatoren
2. Kuhn-Tucker problemer

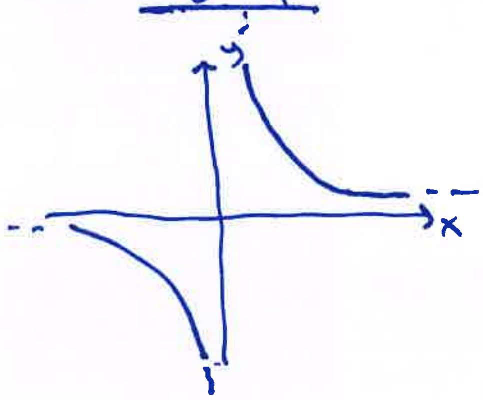
<u>Ex:</u> Opps. $\begin{pmatrix} 1f \\ 2f \end{pmatrix}$ Oppgaveark 30

$x^2y^2 = (xy)^2 = 4^2 = 16$

$\max f(x,y) = x^2y^2 - x^2 - y^2 + 16$ når $\underline{xy = 4}$

$= \max 16 - x^2 - y^2 + 16$ når $xy = 4$

$= \max \underline{32 - x^2 - y^2}$ når $xy = 4$



D: $xy = 4$

$y = 4/x$

<u>ikke kompakt</u>

<u>Lagrange:</u>

<u>Alt I:</u> $L = x^2y^2 - x^2 - y^2 + 16 - \lambda(xy - 4)$

$$\boxed{\begin{aligned} L'_x &= 2xy^2 - 2x - \lambda y = 0 \\ L'_y &= x^2 \cdot 2y - 2y - \lambda x = 0 \\ xy &= 4 \end{aligned}}$$

(1) $2x(y^2-1) = \lambda y \Rightarrow \lambda = \dfrac{2x(y^2-1)}{y}$

(2) $2y(x^2-1) = \lambda x \Rightarrow \lambda = \dfrac{2y(x^2-1)}{x}$

$\Downarrow$

$$\frac{2x(y^2-1)}{y} = \frac{2y(x^2-1)}{x} \quad | \cdot xy$$

$\underline{y=0?}$ $0 = 4$ umulig

$\underline{x=0?}$ $0 = 4$ umulig

$2x^2(y^2-1) = 2y^2(x^2-1)$

$\cancel{2x^2y^2} - 2x^2 = \cancel{2x^2y^2} - 2y^2$

$x^2 = y^2$

$\underline{x = y}$ eller $\underline{x = -y}$

$\underline{x = y}$:

$x^2 = 4$

$x = \pm 2$ $\lambda = 6$

$(2,2;6)$, $(-2,-2;6)$

$f = 16 - 4 - 4 + 16 = 24$

$\underline{x = -y}$:

$-y \cdot y = 4$

$y^2 = -4$

<u>umulig</u>.

Er disse pkt. max?

Innsettingsmetoden: $xy=4 \Rightarrow y = 4/x$

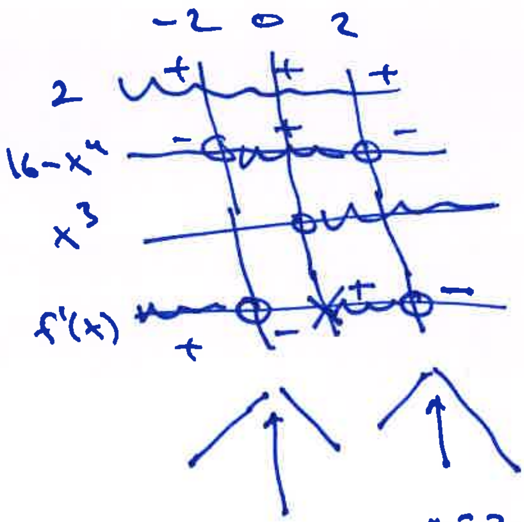
$$f(x,y) = f(x, 4/x) = x^2\cdot(4/x)^2 - x^2 - (4/x)^2 + 16$$
$$= 16 - x^2 - 16/x^2 + 16 = 32 - x^2 - 16/x^2$$

max $f(x) = 32 - x^2 - 16/x^2$

$$f'(x) = -2x - 16\cdot(-2)x^{-3} = -2x + 32/x^3$$
$$= \frac{-2x\cdot x^3}{x^3} + \frac{32}{x^3} = \frac{2(16-x^4)}{x^3} = 0$$

$x^4 = 16$

$x = \pm\sqrt[4]{16} = \pm 2$



$x=-2$
$y=-2$
$f=24$

$x=2$
$y=2$
$f=24$

Konkl: $f_{max} = 24$ i $(2,2), (-2,-2)$ med $\lambda = 6$

Eks: min $f(x,y) = x$ når $y^2 - x^3 = 0$

$$L = x - \lambda \cdot (y^2 - x^3)$$

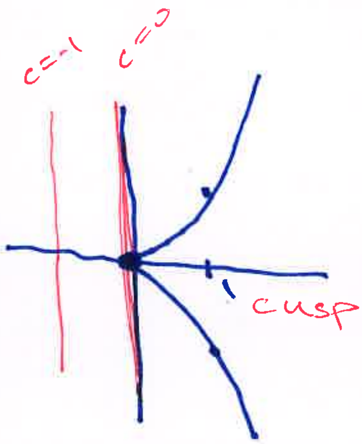
$$\boxed{\begin{aligned} L'_x &= 1 + \lambda \cdot 3x^2 = 0 \\ L'_y &= -\lambda \cdot 2y = 0 \\ & y^2 - x^3 = 0 \end{aligned}}$$

(2) $-\lambda \cdot 2y = 0$

$\lambda = 0$ eller $y = 0$

$\lambda = 0$	$y = 0$
$1 + 0 = 0$	$-x^3 = 0$
umulig	$x = 0$
$\Downarrow$	$1 + \lambda \cdot 0 = 0$
ingen pkt	umulig
	$\Downarrow$
	ingen pkt

ingen ordinære kandidatpkt.



D: $y^2 - x^3 = 0$

$y^2 = x^3$

$x \geq 0, \quad y = \pm\sqrt{x^3}$

$\underline{\underline{f_{min} = 0}}$ i $(0,0)$

Tillatte pkt med degenerert bibetingelse:

$g(x,y) = y^2 - x^3 = 0$

$g'_x = -3x^2 = 0 \qquad x = 0$

$g'_y = 2y = 0 \qquad y = 0$

$y^2 - x^3 = 0 \qquad$ ok

$\Downarrow$

$(0,0)$ er et tillatt pkt med degenert bibetingelse

Nivåkurver for f:

$f(x,y) = \underbrace{x = c}_{\text{vertikal linje}}$

① Tolkning av Lagrange-multiplikatoren λ

Eks: $\max f(x,y) = x^2y^2 - x^2 - y^2 + 16$ når $xy = 4$ ← vi tenker på a som en parameter

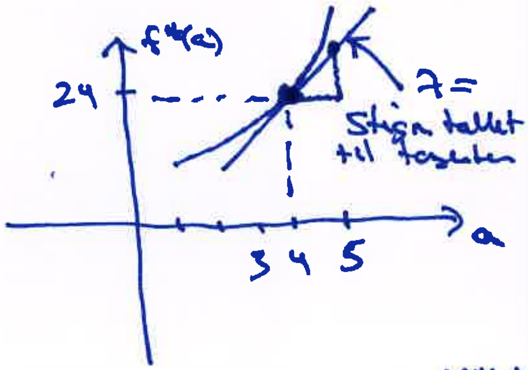
maks.pkt: $(2,2)$; $(-2,-2)$

maks verdi: 24

multiplikator: $\lambda = 6$

Defn: maks.punkt når $g(x,y) = a$: $\left(x^*(a), y^*(a)\right)$

maks.verdi når $g(x,y) = a$: $f^*(a)$ max.verdi-fn.



Tolkning av λ: $\lambda = \dfrac{df^*(a)}{da}$

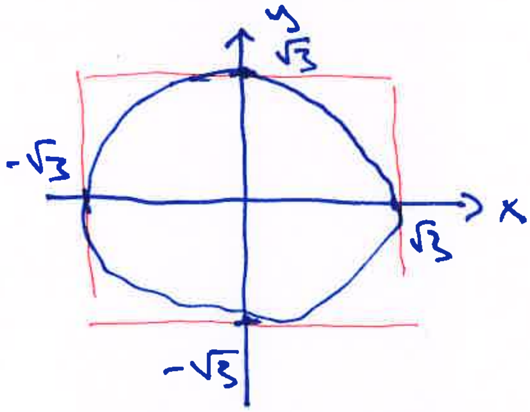
λ = marginale endring i maks. verdi / min. verdi per enhet økning i a, konstanten i bibetingelsen $g(x,y) = \underline{\underline{a}}$

$$f^*(5) \approx f^*(4) + \Delta a \cdot \frac{df^*(a)}{da} = 24 + 1 \cdot 6 = \underline{\underline{30}}$$

Eks: max $f(x,y) = x^2y^2$ når $x^2+y^2+x^2y^2 = 3$

$L = x^2y^2 - \lambda(x^2+y^2+x^2y^2-3)$

$$\boxed{\begin{aligned} L'_x &= 2xy^2 - \lambda(2x+2xy^2) = 0 \\ L'_y &= x^2 2y - \lambda(2y+x^2 2y) = 0 \\ & x^2+y^2+x^2y^2 = 3 \end{aligned}}$$



(1) $2x\cdot(y^2-\lambda-\lambda y^2) = 0$

$x=0$ eller $(\dots)=0$

(2) $2y\cdot(x^2-\lambda-\lambda\cdot x^2) = 0$

$y=0$ eller $(\dots)=0$

$\Downarrow$

(a) $x=0,\ y=0$:

$0+0+0=3$ umulig

(b) $x=0,\ x^2-\lambda-\lambda x^2=0$: $x=0$ $\lambda=0$

$0+y^2+0=3$ $y=\pm\sqrt{3}$

$\rightarrow (0,\pm\sqrt{3};0)$ $f=0$

(c) $y=0,\ y^2-\lambda-\lambda y^2=0$: $y=0$, $\lambda=0$

$x^2+0+0=3$ $x=\pm\sqrt{3}$

$\rightarrow (\pm\sqrt{3},0;0)$ $f=0$

D: $x^2+y^2+x^2y^2=3$

Er D begrenset?

$x=0$: $y^2=3$ $y=\pm\sqrt{3}$

$y=0$: $x^2=3$ $x=\pm\sqrt{3}$

$x^2\le 3$: $-\sqrt{3}\le x\le\sqrt{3}$

$y^2\le 3$: $-\sqrt{3}\le y\le\sqrt{3}$

$\Downarrow$

D er kompakt $\Rightarrow$ det fins makx

(d) $x^2-\lambda-\lambda x^2=0,\ y^2-\lambda-\lambda y^2=0$:

$x^2 = \lambda(1+x^2)$ $\lambda = \dfrac{x^2}{1+x^2}$

$y^2 = \lambda(1+y^2)$ $\lambda = \dfrac{y^2}{1+y^2}$

$\dfrac{x^2}{1+x^2} = \dfrac{y^2}{1+y^2} \Rightarrow x^2(1+y^2) = y^2(1+x^2)$

$x^2 + \cancel{x^2y^2} = y^2 + \cancel{x^2y^2}$

$x^2 = y^2 \Rightarrow x=y$ eller $x=-y$

$\rightarrow (\pm1,\pm1;\tfrac{1}{2})$ $f=1$

$x^2+y^2+x^2y^2=3$

$x^2+x^2+x^2\cdot x^2=3$

$x^4+2x^2-3=0$ annengr. likn. i x^2

$x^2 = \dfrac{-2\pm\sqrt{2^2-4\cdot1\cdot(-3)}}{2}$

$x^2 = \dfrac{-2\pm4}{2} = \cancel{-3},\ 1$

$x=\pm1,\ y=\pm1,\ \lambda=\tfrac{1}{2}$

Tillatte pkt m/degen. bibetingelse:

$$g(x,y) = x^2+y^2+x^2y^2 = 3$$

$$g'_x = 2x + 2x\cdot y^2 = 0 \qquad 2x\cdot(1+y^2) = 0 \qquad \Rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix}$$

$$g'_y = 2y + x^2\cdot 2y = 0 \qquad 2y\,(1+x^2) = 0$$

$$x^2+y^2+x^2y^2 = 3 \qquad 0+0+0 = 3$$

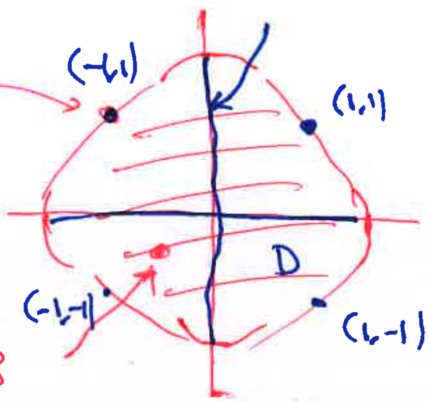
umulig.

Konkl: $f_{max} = 1$ i $(\pm 1, \pm 1)$ med $\lambda = 1/2$

(2) Kuhn-Tucker problem: max/min-problem der bibetingelser = lukkede ulikheter ($\le$, $\ge$)

Ex: $\boxed{\max f(x,y) = x^2y^2 \text{ når } x^2+y^2+x^2y^2 \le 3}$

$x^2+y^2+x^2y^2=3$ randpkt



$x^2+y^2+x^2y^2<3$ indre pkt.

Kandidatpkt:

* randpkt: se utregning for Lagrange-pb. på forrige side

  $f=1$ i $(\pm 1, \pm 1)$ $\lambda = 1/2$

D: kompakt ✓

⇓

det fins max

* indre stasjonære pkt:

$$\left.\begin{matrix} f'_x = 2xy^2 = 0 \\ f'_y = x^2\cdot 2y = 0 \end{matrix}\right\} \begin{matrix} x=0 \\ \text{eller} \\ y=0 \end{matrix} \Rightarrow \begin{matrix} (x,y)=(0,y) & -\sqrt{3}<y<\sqrt{3} \\ (x,y)=(x,0) & -\sqrt{3}<x<\sqrt{3} \end{matrix}$$

$f=0$

Konklusjon: $f_{max} = 1$ i $(\pm 1, \pm 1$