Plan

1. Bestemte integral og antiderivasjon
2. Anvendelser av bestemte integral
3. Uegentlige integral

1 Bestemte integral

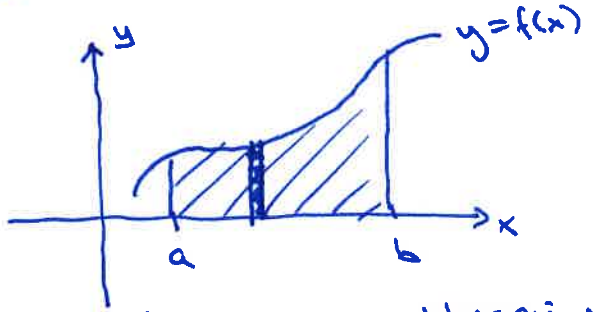
Defn: $\int_a^b f(x)\,dx = F(b) - F(a)$ der $F'(x) = f(x)$

Eks:
$$\int_0^1 x^2\,dx = \underbrace{\left[\tfrac{1}{3}x^3 + C\right]_0^1}_{F(x)} = \left(\tfrac{1}{3}\cdot 1^3 + C\right) - \left(\tfrac{1}{3}\cdot 0^3 + C\right)$$
$$= \tfrac{1}{3} + \cancel{C} - 0 - \cancel{C}$$
$$= \underline{\underline{1/3}}$$

Merk: Et bestemt integral har en bestemt tallverdi.

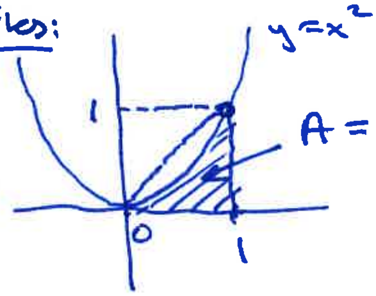
Anta: $f(x)$ er kont. fn. på $[a,b]$ og $f(x) \geq 0$ på $[a,b]$

$$\int_a^b f(x)\,dx = \left\{ \text{arealet til området under grafen til } f \text{ i } [a,b] \right\}$$



Riemannsum = tilnærming til arealet

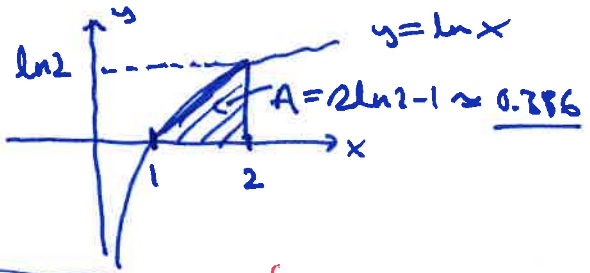
Eks:



$$A = \int_0^1 x^2\,dx$$
$$= \left[\tfrac{1}{3}x^3\right]_0^1$$
$$= \tfrac{1}{3} - 0 = \underline{\underline{1/3}}$$

$A \approx \frac{1\cdot 1}{2} = 1/2$

Eks: $$\int_1^2 \ln x\,dx = \Big[\,x\ln x - x\,\Big]_1^2$$
$$= (2\ln 2 - 2) - (1\cdot \underbrace{\ln 1}_{0} - 1)$$
$$= 2\ln 2 - 2 + 1$$
$$= \underline{2\ln 2 - 1} \simeq 0.386$$



Tilnærming: $A \simeq \dfrac{1\cdot \ln 2}{2} \simeq 0.35$

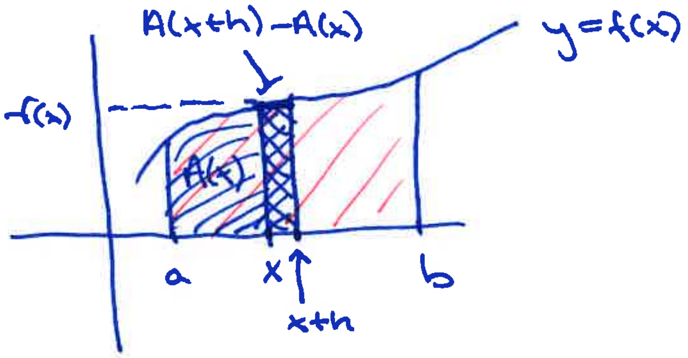
$$\int \ln x\,dx = \int \overset{u'}{1}\cdot \overset{v}{\ln x}\,dx = x\ln x - \int x\cdot \frac{1}{x}\,dx$$
$$= x\ln x - \int 1\,dx$$
$$= \underline{x\ln x - x + C}$$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

Teorem: Hvis $f(x)$ er kont. i $[a,b]$ og $f(x) \geq 0$ for x i $[a,b]$, så er

$$\left\{\begin{array}{l}\text{arealet under grafen} \\ \text{til } f \text{ i intervallet} \\ [a,b]\end{array}\right\} = \int_a^b f(x)\,dx = F(b) - F(a)$$

når $F'(x) = f(x)$.



$A =$ arealet mellom $y=f(x)$ og x-aksen i intervallet $[a,b]$

$$\int_a^b f(x)\,dx = \Big[\,A(x)\,\Big]_a^b$$
$$= A(b) - A(a)$$
$$= A - 0 = A$$

$a \leq x \leq b$:

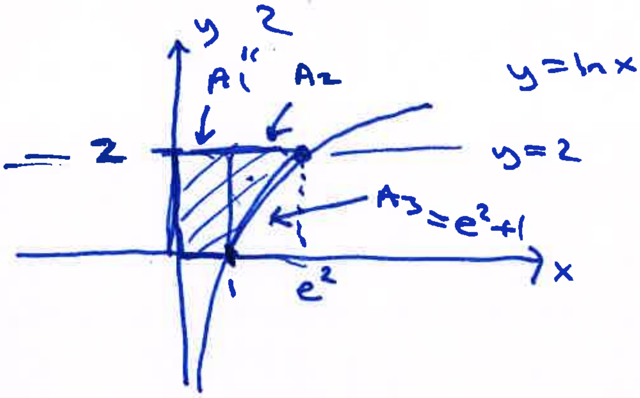
$$A(x) = \left\{\begin{array}{l}\text{arealet mellom } y=f(x) \\ \text{og } x\text{-aksen i} \\ \text{intervallet } [a,x]\end{array}\right\}$$

Fakta: $A(b) = A$
$A(a) = 0$

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}$$
$$\simeq \frac{A(x+h) - A(x)}{h} \quad (h \text{ er liten})$$
$$\simeq \frac{h\cdot f(x)}{h} = f(x)$$

Eks: Finn arealet av området begrenset av

$y = \ln(x)$, $y = 2$, x-aksen og y-aksen.



$$A = A_1 + A_2$$
$$= 1 \cdot 2 + A_2$$
$$= 2 + (e^2-1)\cdot 2 - A_3$$
$$= \cancel{2} + 2e^2 - \cancel{2} - (e^2+1)$$
$$= \underline{\underline{e^2 - 1}} \simeq 6.389$$

$\ln x = 0$ $\quad x = e^0 = \underline{1}$

$\ln x = 2$ $\quad x = e^2$

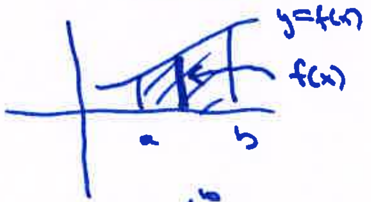
$$A_3 = \int_1^{e^2} \ln x \, dx$$
$$= [x\ln x - x]_1^{e^2}$$
$$= (e^2 \cdot \ln(e^2) - e^2) - (1 \cdot \overset{0}{\ln 1} - 1)$$
$$= e^2 \cdot 2 - e^2 + 1 = \underline{e^2 + 1}$$

$$A_2 = \int_1^{e^2} 2 - \ln x \, dx = [2x - (x\ln x - x)]_1^{e^2}$$
$$= [3x - x\ln x]_1^{e^2} = (3e^2 - e^2 \cdot 2) - (3)$$
$$= \underline{e^2 - 3}$$

(2) Anvendelse av bestemte integral

- areal beregning
- økonomiske anvendelser

i) $\underline{f(x) \geq 0}$:

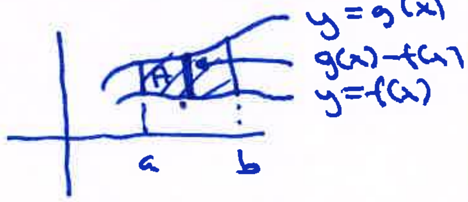


$$A = \int_a^b f(x)\,dx$$

ii) $\underline{f(x) \leq 0}$:

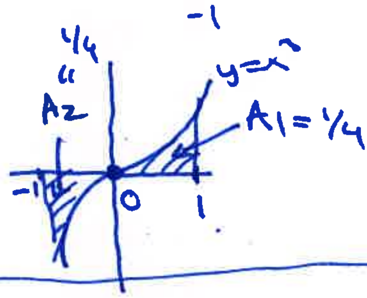
$$A = \int_a^b -f(x)\,dx$$
$$\Updownarrow$$
$$-A = \int_a^b f(x)\,dx$$

iii) $\underline{f(x) \leq g(x)}$:



$$A = \int_a^b g(x) - f(x)\,dx$$

Eks: $\displaystyle\int_{-1}^{1} x^3\,dx = \left[\tfrac{1}{4}x^4\right]_{-1}^{1} = \tfrac{1}{4}\cdot 1^4 - \tfrac{1}{4}(-1)^4 = \tfrac{1}{4} - \tfrac{1}{4} = \underline{\underline{0}}$



Areal: $A_1 = \displaystyle\int_0^1 x^3\,dx = \left[\tfrac{1}{4}x^4\right]_0^1 = \tfrac{1}{4} - 0 = \underline{1/4}$

$A_2 = \displaystyle\int_{-1}^{0} -x^3\,dx = \left[-\frac{x^4}{4}\right]_{-1}^{0}$

$= \left(-0^4/4\right) - \left(-\frac{(-1)^4}{4}\right)$

$= 0 + 1/4 = 1/4$

$A = A_1 + A_2 = 1/4 + 1/4 = \underline{\underline{1/2}}$

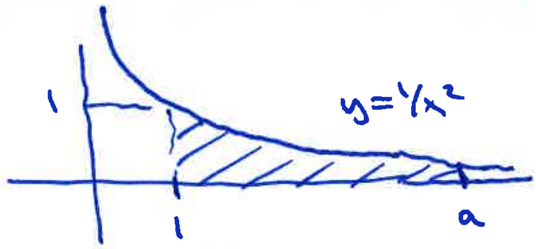
$\displaystyle\int_{-1}^{1} x^3\,dx = \int_{-1}^{0} x^3\,dx + \int_0^1 x^3\,dx$

$= -A_2 + A_1$

$= -\frac{1}{4} + \frac{1}{4} = 0$

③ **Uegentlige integral:**
- grenser $\pm\infty$
- $f(x)$ ikke kont. på $[a,b]$

Eks: $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{a\to\infty}\left(\int_1^a \frac{1}{x^2}\,dx\right) = \lim_{a\to\infty}\left(1 - 1/a\right) = \underline{\underline{1}}$



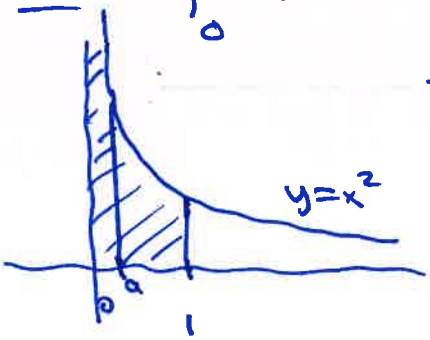
$\displaystyle\int_1^a \underbrace{\frac{1}{x^2}}_{=x^{-2}}\,dx = \left[\frac{x^{-1}}{-1}\right]_1^a = \left[-\frac{1}{x}\right]_1^a$

$= \left(-\frac{1}{a}\right) - \left(-\frac{1}{1}\right)$

$= \underline{1 - 1/a}$

Eks: $\displaystyle\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\left(\int_a^1 \frac{1}{x^2}\,dx\right)$

$= \displaystyle\lim_{a\to 0^+} (1/a - 1) = \underline{\underline{+\infty}}$



$\displaystyle\int_a^1 \frac{1}{x^2}\,dx = \left[-1/x\right]_a^1 = (-1) - (-1/a)$

$= \underline{1/a - 1}$

Ekstra eksempel: Bestemt integral m/ substitusjon

$$\int_0^1 x\sqrt{x^2+1}\,dx =$$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \end{aligned}}$$

Alt I: $x=0 \Rightarrow u = 0^2+1 = 1$, $x=1$: $u = 1^2+1 = 2$

$$= \int_1^2 x\sqrt{u}\cdot\frac{du}{2x} = \int_1^2 u^{1/2}\cdot\frac{1}{2}\,du$$

$$= \left[\frac{1}{2}\cdot\frac{u^{3/2}}{3/2}\right]_{u=1}^{u=2} = \left[\frac{1}{3}u\sqrt{u}\right]_1^2$$

$$= \frac{1}{3}2\sqrt{2} - \frac{1}{3}1\cdot\sqrt{1} = \underline{\underline{\frac{1}{3}(2\sqrt{2}-1)}}$$

Merk: ($\int g(u)\,du$ skal ha grenser i u

derfor $\left\{\begin{array}{l} x=0 \Rightarrow u=1 \\ x=1 \Rightarrow u=2 \end{array}\right\}$

(setter inn i uttrykket for u))

Alt 2: venter med å sette inn grenser

$$= \int_*^* x\cdot\sqrt{u}\,\frac{du}{2\cancel{x}}$$

$$= \int_*^* \frac{1}{2}u^{1/2}\,du$$

$$= \left[\frac{1}{2}\cdot\frac{u^{3/2}}{3/2}\right]_*^*$$

$$= \left[\frac{1}{3}u\sqrt{u}\right]_*^*$$

$$= \left[\frac{1}{3}(x^2+1)\sqrt{x^2+1}\right]_0^1$$

$$= \frac{1}{3}2\cdot\sqrt{2} - \frac{1}{3}1\cdot\sqrt{1}$$

$$= \underline{\underline{\frac{1}{3}(2\sqrt{2}-1)}}$$

$*$: venter med å sette inn grenser til vi har byttet tilbake til x