Plan

1 Repetisjon Utvalgte temaer
2 Gjennomgang Oppgaver

① Eksamensoppgaver

Eksamen 06/2020

6c max $f(x,y) = xy$ når $4x^2 - 24x + 16y^2 = 64$

Lagrangeproblem:

Kandidatpkt:

$$\mathcal{L} = xy - \lambda(4x^2 - 24x + 16y^2 - 64)$$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= y - \lambda(8x - 24) = 0 \\ \mathcal{L}'_y &= x - \lambda(32y) = 0 \\ &4x^2 - 24x + 16y^2 = 64 \end{aligned}}$$

Lagrangebetingelsene

(1) $y = \lambda \cdot (8x - 24)$

$\lambda = \dfrac{y}{8x-24}$

(2) $x = (32y)\lambda$

$\lambda = \dfrac{x}{32y}$

Sjekk:

$\cancel{8x - 24 = 0}$
$\cancel{x = 3,\ y = 0}$
~~umulig~~

$\cancel{32y = 0}$
$\cancel{y = 0,\ x = 0}$
$\cancel{\lambda = 0}$ ~~umulig~~

$\dfrac{y}{8x-24} = \dfrac{x}{32y} \quad | \cdot (8x-24) \cdot 32y$

$32y^2 = 8x^2 - 24x \quad | : 2 \qquad 16y^2 = 4x^2 - 12x$

(3) $4x^2 - 24x + 4x^2 - 12x = 64$

$8x^2 - 36x - 64 = 0 \quad | : 4$

$2x^2 - 9x - 16 = 0$

$x = \dfrac{9 \pm \sqrt{81 - 4 \cdot 2 \cdot (-16)}}{2 \cdot 2} = \dfrac{9 \pm \sqrt{209}}{4}$

$x_1 \simeq \underline{5.864}$, $x_2 \simeq \underline{-1.364}$

$y^2 = \dfrac{4x^2 - 12x}{16} = \dfrac{x^2 - 3x}{4}$

$y_1 \simeq \underline{\pm 2.049}$ $\quad y_2 = \underline{\pm 1.220}$

$\Downarrow$

fire kandidatpkt:

$(x,y) \simeq (5.864, 2.049) \leftarrow$ beste kandidat $f \approx \underline{12.02}$
$(5.864, -2.049)$
$(-1.364, 1.220)$
$(-1.364, -1.220)$

Angir om beste kandidatpkt er maks:

D: $\underbrace{4x^2 - 24x + 16y^2}_{g(x,y)} = 64$ ellipse (fra a) med $t=4$) C

$\Downarrow$ EVS

Kompakt $\Rightarrow$ problemet har et maks

ingen unntakspkt (alle pkt på ellipsen har entydig tangent)

$\Downarrow$

$f_{max} \cong f(5.864, 2.049)$

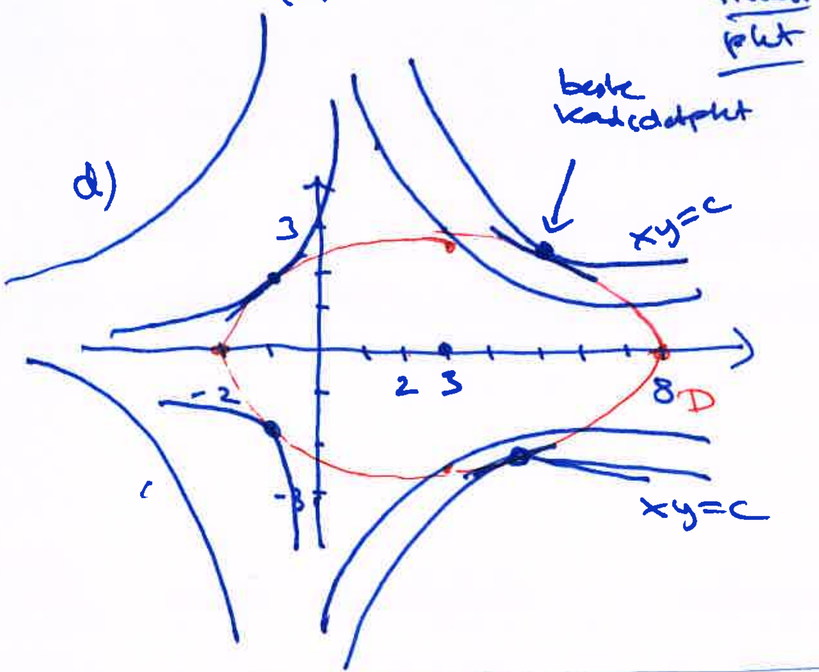
$\approx \underline{\underline{12.02}}$

$g'_x = 8x - 24 = 0$ $x=3$

$g'_y = 32y = 0$ $y=0$

$4x^2 - 24x + 16y^2 = 64$ ikke tillatt pkt

d)



(fra a)

D: ellipse, sentr (3,0) halvakser $\underline{a=5}$, $\underline{b=5/...}$

$f(x,y) = xy$

Nivåkurver: $f(x,y) = c$

$xy = c$

$y = c/x$

Fra teori vet vi at løsninger av Lagrange-betingelsene = pkt der ellipsen møter en nivåkurve for f i en tangent.

Eksamen 05/2019

5. $f(x,y) = y^2 - x^3 + 3x$

C: nivåkurven til f gjennom $(-1,2)$

$f(-1,2) = 2 \Rightarrow \boxed{C : f(x,y) = 2}$

a) $f'_x = -3x^2 + 3 = 0 \quad x^2 = 1 \quad x = \pm 1$

$f'_y = 2y = 0 \quad y = 0$

Stasjonære pkt $(1,0)$, $(-1,0)$

$$H(f) = \begin{pmatrix} -6x & 0 \\ 0 & 2 \end{pmatrix}$$

$$H(f)(1,0) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix}$$

$\det = -12 < 0$

$(1,0)$ sadelpkt

$$H(f)(-1,0) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$\det = 12 > 0$

$A = 6 > 0 \quad tr = A + C = 6 + 2 = 8$

$(-1,0)$ lokalt min

b) Tangent: $(-1,2)$

$y - y_0 = a \cdot (x - x_0)$

$y - 2 = 0\,(x+1) = 0 \Rightarrow y = 2$

$$y' = -\frac{f'_x}{f'_y} = -\frac{-3x^2+3}{2y} = \frac{3x^2-3}{2y}$$

$$y'(-1,2) = \frac{3\cdot(-1)^2 - 3}{2\cdot 2} = 0$$

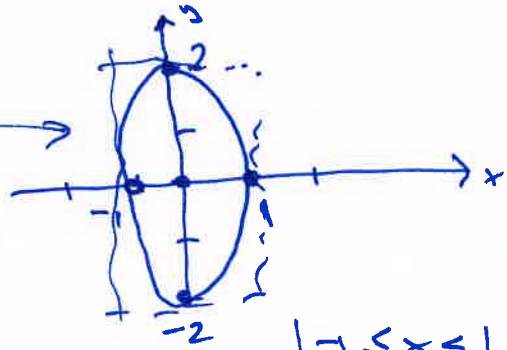
c) $\frac{4x^2}{4} + \frac{y^2}{4} = \frac{4}{4}$ ellipse

sentr $(0,0)$

$a = 1$

$b = 2$

$$\frac{x^2}{1} + \frac{y^2}{4} = 1$$



Ja, begrenset siden $\begin{cases} -1 \le x \le 1 \\ -2 \le y \le 2 \end{cases}$

d) max $f(x,y) = y^2 - x^3 + 3x$ når $4x^2 + y^2 = 4$

D: ellipse (kompakt)
⇓ EVS
det fins et maks,
ingen randpunkt (tenkt i alle pkt av en ellipse)

$L = y^2 - x^3 + 3x - \lambda(4x^2 + y^2 - 4)$

$$\boxed{\begin{aligned} L'_x &= -3x^2 + 3 - \lambda \cdot (8x) = 0 \\ L'_y &= 2y - \lambda(2y) = 0 \\ & 4x^2 + y^2 = 4 \end{aligned}}$$

(2) $2y(1-\lambda) = 0$

$\underline{y=0}$ eller $\underline{\lambda = 1}$

$4x^2 = 4$

$x^2 = 1$

$\underline{y=0}$, $\underline{x = \pm 1}$

(1) $0 - \lambda \cdot 8(\pm 1) = 0$

$\underline{\lambda = 0}$

$(x,y;\lambda) = \underline{(\pm 1, 0; 0)}$

$f(1,0) = 0^2 - 1 + 3 = \underline{2}$

$f(-1,0) = 0^2 + 1 - 3 = \underline{-2}$

(1) $-3x^2 + 3 - 8x = 0$

$3x^2 + 8x - 3 = 0$

$x = \dfrac{-8 \pm \sqrt{64 - 4 \cdot 3(-3)}}{2 \cdot 3} = \dfrac{-8 \pm 10}{6}$

$x = 2/6 = \underline{1/3}$ eller $x = \underline{-3}$

$4 \cdot (1/3)^2 + y^2 = 4$

$4/9 + y^2 = 4$

$y^2 = 4 - 4/9 = 32/9$

$y = \pm \sqrt{32}/3$

$4 \cdot (-3)^2 + y^2 = 4$

$36 + y^2 = 4$

$y^2 = -32$

ingen løsn.

$\underline{(1/3, +\sqrt{32}/3; 1)}$, $\underline{(1/3, -\sqrt{32}/3; 1)}$

$f(1/3, \pm\sqrt{32}/3) = \frac{32}{9} - \frac{1}{27} + 1$

$= \dfrac{32 \cdot 3 - 1 + 27}{27}$

$= \dfrac{122}{27} > 2$

$\underline{\text{Konkl.}}$ $f_{max} = \underline{\underline{\dfrac{122}{27}}}$; $\underline{\underline{(1/3, \pm\sqrt{32}/3; 1)}}$

6. min x når $y^2 - x^3 + 3x = 2$

$D = ?$

$y^2 = x^3 - 3x + 2$

Fortegnslinje for $h(x) = x^3 - 3x + 2$

$x^3 - 3x + 2 = 0 \qquad x = \pm 1,\ x = \pm 2$

$x = 1$ er løsn.

$x^3 - 3x + 2 = (x-1)\cdot(x^2 + x - 2)$

~~$x^2 + x + 2 = 0$~~

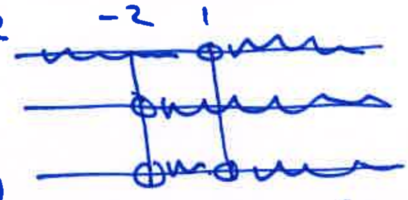
~~$x = \frac{-1 \pm \sqrt{1 - 4\cdot 2}}{2}$ ingen løsn.~~

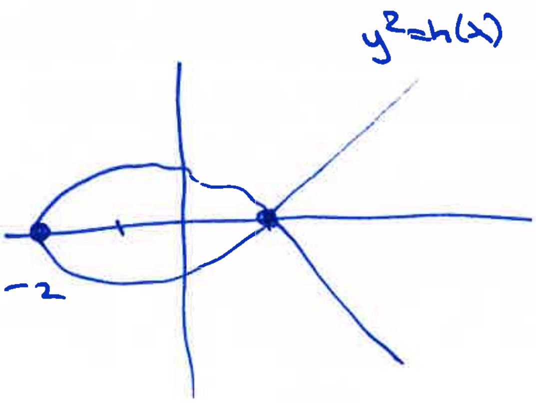
$x^2 + x - 2 = 0$

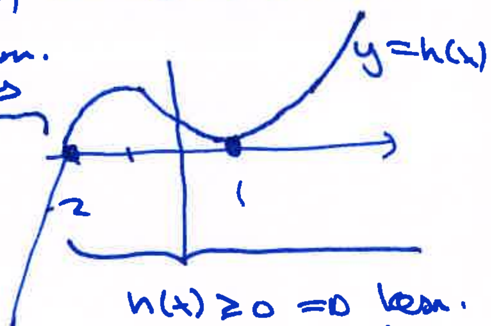
$x = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$

$x^3 - 3x + 2 = (x-1)(x^2 + x - 2) = (x-1)(x-1)(x+2)$

$$\boxed{y^2 = (x-1)^2(x+2)}$$







$h(x) \geq 0 \Rightarrow$ løsn. for y

min x når $y^2 = h(x) \Rightarrow x_{min} = -2$ i $(-2, 0)$

Eksamen 12/2019

Sab $f(x,y) = 4x^2 + 9y^2 - x^2y^2$

a) $f'_x = 8x - 2xy^2 = 0$ $\quad 2x(4-y^2) = 0$

$f'_y = 18y - 2x^2y = 0$ $\quad 2y(9-x^2) = 0$

Stasjonære pkt:

$x = 0$ eller $y^2 = 4$

$y = 0$ eller $x^2 = 9$

$\Rightarrow$ i) $x = y = 0 \Rightarrow (0,0)$

ii) $x = \pm 2, y = \pm 3$

$\Rightarrow (\pm 2, \pm 3)$

b) $H(f) = \begin{pmatrix} 8-2y^2 & -4xy \\ -4xy & 18-2x^2 \end{pmatrix}$

$H(f)(0,0) = \begin{pmatrix} 8 & 0 \\ 0 & 18 \end{pmatrix}$ $\quad$ det $= 8 \cdot 18 > 0$, tr $= 8 + 18 > 0$

(0,0) lokalt min

$H(f)(\pm 2, \pm 3) = \begin{pmatrix} -10 & -24 \\ -24 & 10 \end{pmatrix}$ $\quad$ det $= -10 \cdot 10 - (-24)^2 < 0$

$(\pm 2, \pm 3)$ Sadelpkt

2 Repetisjon

(a) Noen viktige kurver i planet

Ellipse: $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$ $\begin{cases} \text{senter } (x_0, y_0) \\ \text{halvakser } a, b \end{cases}$

Sirkel: $(x-x_0)^2 + (y-y_0)^2 = r^2$ $\begin{cases} \text{senter i } (x_0, y_0) \\ \text{radius } r \end{cases}$

$x^2 + 4y^2 = 36$ ellipse $\qquad$ $2x^2 + 2y^2 = 18$ sirkel

Hyperbel: $(x-a)(y-b) = c \Leftrightarrow y - b = \frac{c}{x-a}$

$$y = b + \frac{c}{x-a} = \frac{b(x-a)+c}{x-a}$$

Parabler: $y = a \cdot (x - x_0)^2$

Linjer: $y - y_0 = a \cdot (x - x_0)$

(b) Kontantstrømmer

$$120^{0} + 120 \cdot 1{,}04^{1} + 120 \cdot 1{,}04^{2} + \ldots + 1{,}04^{24}$$

$$= a_1 \cdot \frac{(1+r)^n - 1}{(1+r) - 1} = 120 \cdot \frac{1{,}04^{25} - 1}{0.04}$$

$k = 1 + r$ koeff. i den geometriske rekken $= 1{,}04$

a_1 = første ledd $= 120$

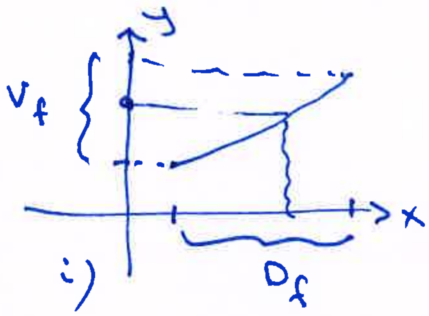
n = antall ledd $= 25$

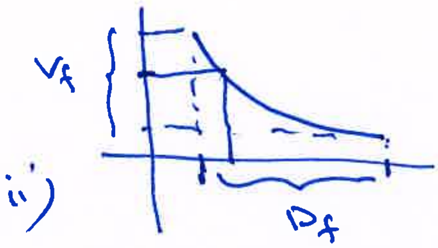
c) Omvendte funksjoner

$f(x)$ med defn. mengde D_f

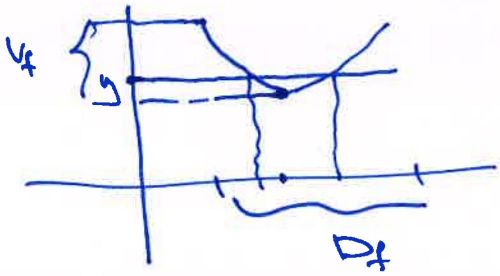
Eksistens: i) $f(x)$ voksende i hele D_f $\Rightarrow$ f^{-1} fins

ii) $f(x)$ avtagende i —"— $\Rightarrow$ f^{-1} fins





iii) Hvis f er kont. og bytter mellom voksende og avtagende i et indre pkt i D_f $\Rightarrow$ f^{-1} fins ikke



f^{-1} fins ikke (ikke entydig verdi)

Finne f^{-1} eksplisitt:

$y = f(x)$ ← løs for x

$y = x \cdot e^x, \quad x \geq -1$

$$\boxed{(f^{-1})'(f(x)) = \frac{1}{f'(x)}}$$

d) Delbrøk:

Eks: $\dfrac{2x-1}{x^2-4} = \dfrac{A}{x-2} + \dfrac{B}{x+2}$ $\quad A = 3/4, \quad B = 5/4$

- Kun aktuelt om nevner har grad ≥ 2
- bruk polynomdiv. først hvis grad teller ≥ grad nevner

$2x-1 = A(x+2) + B(x-2)$

$x=-2: \; -5 = -4B$

$\underline{B = 5/4}$

$x = 2: \; 3 = 4A$

$\underline{A = 3/4}$

$2x-1 = \underbrace{(A+B)}_{=2}x + \underbrace{(2A-2B)}_{=-1}$

e) Inverse matriser

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$A^{-1} = \dfrac{1}{|A|} \cdot \operatorname{adj}(A)$ hvis $|A| \neq 0$

$= \dfrac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$

$(-1)^{1+1}$

$C_{11} = +1 \cdot (2 \cdot 1 - 4(-1))$

$= \underline{+6}$

$= \dfrac{1}{6} \cdot \begin{pmatrix} 6 & 3 & -3 \\ -2 & 0 & 2 \\ 2 & -3 & 1 \end{pmatrix}^T$

$C_{21} = -(1+1) = \underline{-2}$

$C_{31} = +(4-2) = 2$

$= \dfrac{1}{6} \begin{pmatrix} 6 & -2 & 2 \\ 3 & 0 & -3 \\ -3 & 2 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \end{pmatrix}$

$|A| = 1 \cdot 6 + 1 \cdot 3 + 1 \cdot (-3) = 6$