Plan

1. Forklaring av Lagranges multiplikatormetode
2. Tolkning av Lagrangemultiplikatorer

Ex: max $f(x,y) = x^2y^2$ når $\underbrace{x^2+y^2+x^2y^2 = 3}$

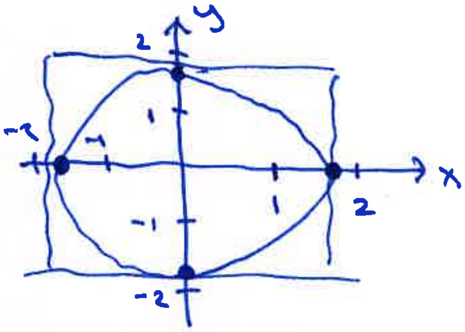
$D: x^2+y^2+x^2y^2=3$

$$y^2+x^2y^2 = 3-x^2$$
$$y^2(1+x^2) = 3-x^2$$
$$y^2 = \frac{3-x^2}{1+x^2}$$

Må ha $3-x^2 \geq 0$

$$x^2 \leq 3$$
$$-\sqrt{3} \leq x \leq \sqrt{3}$$

$$y = \pm\sqrt{\frac{3-x^2}{1+x^2}}, \quad -\sqrt{3}\leq x \leq \sqrt{3}$$



D er kompakt (lukket, begrenset)

begrenset fordi $-\sqrt{3} \leq x \leq \sqrt{3}$

$-\sqrt{3} \leq y \leq \sqrt{3}$

Merk at ~~[illegible]~~

$x^2 \leq 3$, $y^2 \leq 3$

$\Downarrow$ EVS = ekstremverdisetn.

problemet har et maksimum

Eivind Eriksen, Kontor B4-032, eivind.eriksen@bi.no
http://www.dr-eriksen.no/teaching/MET1180/

① Forklaring av Lagranges multiplikatormetode

$$\max\ f(x,y) = x^2y^2 \quad \text{når} \quad \overbrace{x^2+y^2+x^2y^2}^{g(x,y)} = \overbrace{3}^{a}$$

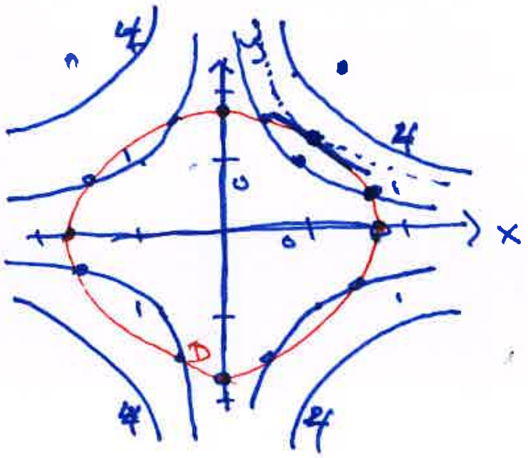
$$\mathcal{L} = x^2y^2 - \lambda(x^2+y^2+x^2y^2-3) = f(x,y) - \lambda(g(x,y)-a)$$

$$\text{FOC}\begin{cases} \mathcal{L}'_x = 2xy^2 - \lambda(2x+2xy^2) = 0 \\ \mathcal{L}'_y = 2x^2y - \lambda(2y+2x^2y) = 0 \end{cases}$$

$$\text{C}\ \{\ x^2+y^2+x^2y^2 = 3$$

Lagrangebetingelsene = FOC + C

Løsninger $(x,y;\lambda)$ = kandidater for maks.



D = tillatte pkt
$x^2+y^2+x^2y^2 = 3$

$f(x,y) = x^2y^2$:

Nivåkurver for f: $f(x,y) = c$
$x^2y^2 = c$

c > 0:
$y^2 = c/x^2$
$y = \pm\sqrt{c/x^2}$
$y = \pm\sqrt{c}/x$
hyperbler

c = 1: $y = \pm 1/x$
c = 4: $y = \pm 2/x$

c = 0:
$x^2y^2 = 0$
$x = 0$ eller $y = 0$
aksene

c < 0:
$x^2y^2 = c$
ingen pkt

Lagranges multiplikatormetode

Hvis (x^*,y^*) er et maks, så må ~~[illegible]~~ D: $x^2+y^2+x^2y^2 = 3$ (rød kurve) skjære nivåkurven til f i (x^*,y^*) i en tangent.

$$\text{FOC:}\begin{cases} \mathcal{L}'_x = f'_x - \lambda g'_x = 0 \\ \mathcal{L}'_y = f'_y - \lambda g'_y = 0 \end{cases} \iff \lambda = \boxed{\frac{f'_x}{g'_x} = \frac{f'_y}{g'_y}} \implies \frac{f'_x \cdot g'_y}{f'_y \cdot g'_y} = \frac{f'_y \cdot g'_x}{f'_y \cdot g'_y}$$

$$\iff \frac{f'_x}{f'_y} = \frac{g'_x}{g'_y}$$

$$-\frac{f'_x}{f'_y} = -\frac{g'_x}{g'_y}$$

tangenter er like

Konklusjon:

FOC: $\boxed{\begin{aligned} L'_x &= f'_x - \lambda g'_x = 0 \\ L'_y &= f'_y - \lambda g'_y = 0 \end{aligned}} \iff \boxed{-\dfrac{f'_x}{f'_y} = -\dfrac{g'_x}{g'_y}}$

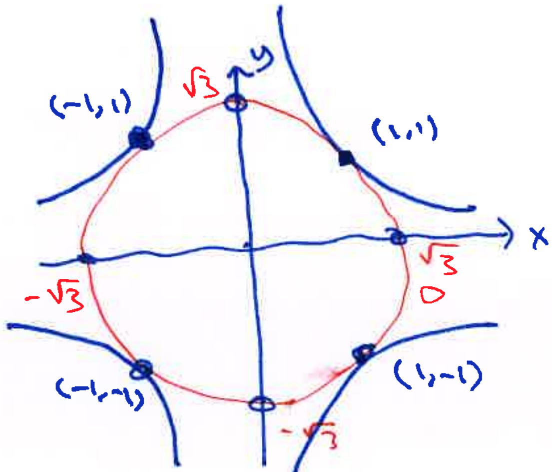
Stigningstallet til tangenten for $f(x,y)=c$ (blå kurve)

Stigningstallet til tangenten for $g(x,y)=a$ (D, red kurve)

+ C: $\boxed{g(x,y)=a}$

Eks: max $f = x^2y^2$ når $x^2+y^2+x^2y^2=3$

$L = x^2y^2 - \lambda(x^2+y^2+x^2y^2-3)$



FOC: $L'_x = 2xy^2 - \lambda(2x+2xy^2) = 0$

$L'_y = 2x^2y - \lambda(2y+2x^2y) = 0$

C: $x^2+y^2+x^2y^2 = 3$

i) $2x(y^2-\lambda(1+y^2)) = 0$ $\quad x=0$ eller $y^2 = \lambda(1+y^2)$

ii) $2y(x^2-\lambda(1+x^2)) = 0$ $\quad y=0$ eller $x^2 = \lambda(1+x^2)$

a) $x=0, y=0$: $0+0+0=3 \Rightarrow$ ingen løsn.

b) $x=0, x^2=\lambda(1+x^2)$: $\lambda=0, x=0, 0+y^2+0=3$; $y^2=3 \Rightarrow y=\pm\sqrt{3}$ $\Big\}$ $(0,\pm\sqrt{3}; 0)$ $f=0$

c) $y^2=\lambda(1+y^2), y=0$: $\lambda=0, y=0, x^2+0+0=3$; $x^2=3 \Rightarrow x=\pm\sqrt{3}$ $\Big\}$ $(\pm\sqrt{3}, 0; 0)$ $f=0$

d) $y^2=\lambda(1+y^2), x^2=\lambda(1+x^2)$:

$\lambda = \dfrac{y^2}{1+y^2} = \dfrac{x^2}{1+x^2}$ $\quad (1+x^2, 1+y^2 > 0)$

$y^2(1+x^2) = x^2(1+y^2)$

$y^2 + \cancel{x^2y^2} = x^2 + \cancel{x^2y^2}$

$y^2 = x^2$ $\quad y=\pm x$

$x^2 + x^2 + x^2 \cdot x^2 = 3$

$x^4 + 2x^2 - 3 = 0$

$u=x^2$: $u^2+2u-3=0$

$(u+3)(u-1)=0$

$\cancel{u=x^2=-3}$ eller $u=x^2=1$

$\Rightarrow x=\pm 1$

$(\pm1, \pm1; 1/2)$ $f=1$

Sjekk evt tillatte pkt med degenerert bibetingelse

$$\underbrace{x^2+y^2+x^2y^2}_{g(x,y)} = \underbrace{3}_{a}$$

$$\begin{aligned} g'_x &= 0 & 2x+2xy^2 &= 0 & 2x(1+y^2) &= 0 & x &= 0 \\ g'_y &= 0 & 2y+2x^2y &= 0 & 2y(1+x^2) &= 0 & y &= 0 \\ g &= a & x^2+y^2+x^2y^2 &= 3 \end{aligned}$$

$\Downarrow$

$(0,0)$

degenerert bibetingelse

ikke tillatt

$x^2+y^2+x^2y^2=0 \neq 3$

Konkl: ingen unntakspkt

$\Downarrow$

$f_{max} = \underline{\underline{1}}$ i $(\pm1, \pm1)$ med $\lambda = 1/2$

② Tolkning av Lagrangemultiplikatoren λ

Eks: $\boxed{\max\ f(x,y) = x^2y^2 \text{ når } x^2+y^2+x^2y^2 = a}$

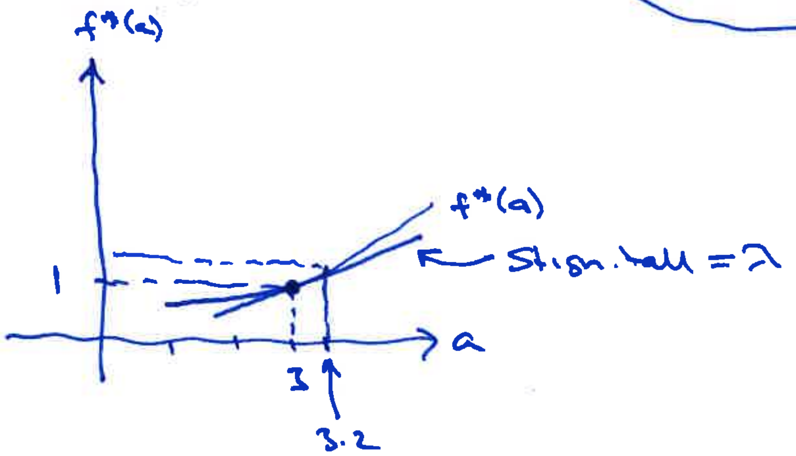
Lagrangeproblem med parameter a = konstanten i bibetingelsen

Vi kaller maks. verdien $f^*(a)$.

Vi vet at $f^*(3) = 1$.

($a=3$ gir $f_{max}=1$)

Tolkning av λ: $\lambda = \dfrac{df^*(a)}{da}$ = marginal endring i maks.verdi når vi endrer a = konstant i bibetingelsen



For eksempel:

max $f = x^2y^2$ når $x^2+y^2+x^2y^2 = 3.2$

$\Downarrow$

$$f_{max} = f^*(3.2) \cancel{+\Delta a} \simeq \underbrace{f^*(3)}_{=1} + \underbrace{\Delta a}_{(3.2-3)} \cdot \underbrace{\frac{df^*(a)}{da}}_{\lambda}$$

$f^*(3.2) \simeq 1 + 0.2 \cdot 1/2 = \underline{\underline{1.1}}$

Eivind Eriksen, Kontor B4-032, eivind.eriksen@bi.no
http://www.dr-eriksen.no/teaching/MET1180/

Del 2: Oppgaveark 31

5. C: $y(x^2+y^2) = 2(x^2-y^2)$

a) $\underline{y=-1}$:

$$-1(x^2+1) = 2(x^2-1)$$
$$-x^2-1 = 2x^2-2$$
$$1 = 3x^2$$
$$x^2 = 1/3 \qquad x = \underline{\pm\sqrt{1/3}}$$

Svar: $\underline{\underline{(\pm\sqrt{1/3}, -1)}}$ med $y=-1$

b) $\underbrace{y(x^2+y^2)-2(x^2-y^2)}_{g(x,y)} = \underset{a}{0}$

$$y' = -\frac{g'_x}{g'_y}$$

$$g'_x = y(2x)-2(2x) = \underline{2xy-4x}$$
$$g'_y = \underline{x^2+3y^2+4y}$$

$$y' = -\frac{2x(y-2)}{x^2+3y^2+4y}$$

$(\sqrt{1/3},-1)$: $\quad y' = -\frac{2\sqrt{1/3}(-3)}{1/3+3-4} = \frac{6\sqrt{1/3}}{-2/3} = \underline{-9\cdot\sqrt{1/3}} = -9\cdot\sqrt{\frac{1\cdot 3}{3\cdot 3}}$

$$= \frac{-9\sqrt{3}}{3} = \underline{-3\sqrt{3}}$$

$$y+1 = -3\sqrt{3}\left(x-\sqrt{1/3}\right)$$
$$y+1 = -3\sqrt{3}x+3$$
$$y = \underline{-3\sqrt{3}x+2}$$

$(-\sqrt{1/3},-1)$: $\quad y' = -\frac{2(-\sqrt{1/3})(-3)}{1/3+3-4} = \underline{3\sqrt{3}}$

$$y+1 = 3\sqrt{3}\left(x+\sqrt{1/3}\right) = 3\sqrt{3}x+3$$
$$y = \underline{3\sqrt{3}x+2}$$

c) max/min. $f(x,y) = y$ når $y(x^2+y^2)-2(x^2-y^2)=0$

$$\mathcal{L} = y - \lambda\left(y(x^2+y^2) - 2(x^2-y^2)\right)$$

$$\boxed{\begin{aligned} L'_x &= -\lambda(2xy-4x)=0 \\ L'_y &= 1-\lambda(x^2+3y^2+4y)=0 \\ & y(x^2+y^2)-2(x^2-y^2)=0 \end{aligned}}$$

$-\lambda \cdot 2x(y-2)=0$

$\lambda=0$ eller $x=0$ eller $y=2$

$1-0=0$ $\Downarrow$ ingen kand.

$y \cdot y^2-2(-y^2)=0$
$y^3+2y^2=0$
$y^2(y+2)=0$
$y=0$ eller $y=-2$

$x=0$, $y=0$: $1-\lambda \cdot 0=0$ $\Downarrow$ ingen kand

$x=0$, $y=-2$: $1-\lambda(4)=0$, $\lambda=1/4$ $\Downarrow$ $(0,-2;1/4)$, $f=-2$

$y=2$: $2\cdot(\cancel{x^2}+4)-2(\cancel{x^2}-4)=0$
$16=0$
umulig
$\Downarrow$
ingen kand.

Avgjøre om $f(0,-2)=-2$ er maks/min:

$y=a$: $a(x^2+a^2)-2(x^2-a^2)=0$
$ax^2-2x^2+a^3+2a^2=0$
$(a-2)x^2+a^3+2a^2=0$
$(a-2)x^2=-(a^3+2a^2)=-a^2(a+2)$ $\quad (: a-2)$
$x^2=-\dfrac{a^2(a+2)}{a-2}$

For hvilke verdier av a har bibetingelsen løsn. for x

	-2	0	2
$-a^2$	$-$	$-$ 0 $-$	$-$
$a+2$	$-$ 0	$+$	$+$
$a-2$	$-$	$-$	0 $+$
HS	$-$ 0	$+$ 0 $+$	$\times$ $-$
≥ 0	$[$		$)$

$\Downarrow$

Mulige verdier for $f(x,y)=y$ når (x,y) er med i D:

$[-2,2)$

Dvs: f_{max} finnes ikke
$f_{min}=\underline{\underline{-2}}$

6. max/min $f(x,y) = x^3 + 3xy + y^3$ når $xy = 1$

$=$ max/min $x^3 + 3 + y^3$ når $xy = 1$

$\mathcal{L} = x^3 + 3 + y^3 - \lambda(xy - 1)$

$\boxed{\begin{aligned} \mathcal{L}'_x &= 3x^2 - \lambda y = 0 \\ \mathcal{L}'_y &= 3y^2 - \lambda x = 0 \\ xy &= 1 \end{aligned}}$

$\Rightarrow \lambda y = 3x^2 \Rightarrow \lambda = \dfrac{3x^2}{y}$ ✓

$\lambda x = 3y^2 \Rightarrow \lambda = \dfrac{3y^2}{x}$

$\uparrow$

$x \neq 0,\ y \neq 0$

$\dfrac{3x^2}{y} = \dfrac{3y^2}{x} \quad | \cdot xy$

$3x^2 \cdot x = 3y^2 \cdot y \quad | : 3$

$x^3 = y^3 \quad | \sqrt[3]{\ }$

$x = y$

$y \cdot y = 1 \quad y^2 = 1 \quad y = \pm 1$

Unntakspkt:

$g = xy = 1$

$\left.\begin{aligned} g'_x &= y = 0 \\ g'_y &= x = 0 \\ xy &= 1 \end{aligned}\right\}$ Ingen unntakspkt

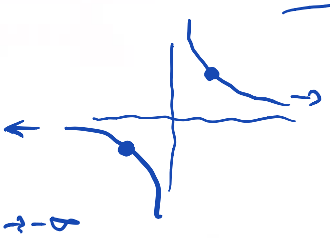
Ord. kandidatpkt: $(1,1;3)$, $(-1,-1;-3)$

$f = 5$ $\qquad f = 1$

max? **Nei** $\qquad$ min? **Nei**

$f(x,y) = x^3 + 3xy + y^3$ når $xy = 1$

$= x^3 + 3 + y^3$ — " —



$x \to \infty$

$y = 1/x \to 0$

$f \to \infty$

$f(2, 1/2) = 8 + 3 + 1/8 > 5$

$x \to -\infty$

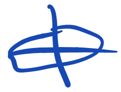
$y = 1/x \to 0$

$f \to -\infty$

$f(-2, -1/2) = -8 + 3 - 1/8$

$= -5 - 1/8 < 1$

4. max/min $f = xy$ når $x^2+y^2=4$



kompakt $\Rightarrow$ EVS $\Rightarrow$ fins maks/min

Ingen unntakspkt:

$x^2+y^2=4 \leftarrow$ nei

$\left.\begin{array}{l} g'_x = 2x = 0 \\ g'_y = 2y = 0 \end{array}\right\} \Rightarrow (x,y)=(0,0)$

$$L = xy - \lambda(x^2+y^2-4)$$

$$\boxed{\begin{array}{l} L'_x = y - \lambda\cdot 2x = 0 \\ L'_y = x - \lambda \cdot 2y = 0 \\ x^2+y^2=4 \end{array}}$$

$y = 2\lambda x$

$x - 2\lambda(2\lambda x) = 0$

$x(1-4\lambda^2) = 0$

$\cancel{x=0}$ eller $\lambda = 1/2$ eller $\lambda = -1/2$

$\cancel{y=0}$ umulig

$\lambda = 1/2$:

$y = x$

$2x^2 = 4$

$x^2 = 2$

$x = y = \sqrt{2}$ eller $x = y = -\sqrt{2}$

$(\sqrt{2}, \sqrt{2}; 1/2)$ $f=2$

$(-\sqrt{2}, -\sqrt{2}; 1/2)$ $f=2$

$\lambda = -1/2$:

$y = -x$

$2x^2 = 4$

$x^2 = 2$

$x = \pm\sqrt{2}$

$(\sqrt{2}, -\sqrt{2}; -1/2)$ $f=-2$

$(-\sqrt{2}, \sqrt{2}; 1/2)$ $f=-2$

Konkl.:

$f_{max} = 2$ $\quad$ $f_{min} = -2$