Plan

① Funksjoner i to variabler

Eks:

$f(x,y) = 2x + 3y$ lineært

$f(x,y) = x^2 + y^2$ polynom

$f(x,y) = \dfrac{x+y}{x-y}$ rasjonalt uttrykk

$f(x) = x^2 - 2x + 3$

$y = f(x)$

Skriver: $z = f(x,y)$

Generelt: $f(x,y) = \underbrace{\text{uttrykk i } x \text{ og } y}_{\text{funksjonsuttrykket til } f}$

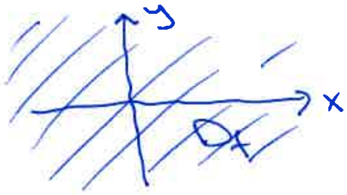
Eks: $f(x,y) = x^2 + y^2$

(x,y)	$z = f(x,y) = x^2 + y^2$
$(0,0)$	$f(0,0) = 0$
$(1,0)$	$f(1,0) = 1^2 + 0^2 = 1$
$(0,1)$	$f(0,1) = 0^2 + 1^2 = 1$
$(-1,0)$	$f(-1,0) = (-1)^2 + 0^2 = 1$
$\vdots$	$\vdots$

Definisjonsområde: D_f

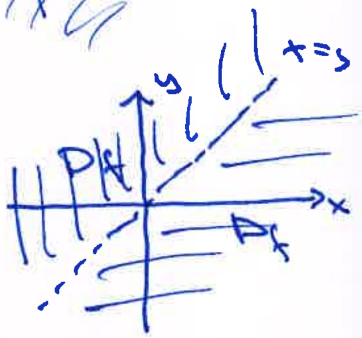
D_f = alle tallpar (x,y) som vi kan sette inn i funksjon $f(x,y)$

Eks: $f(x,y) = x^2 + y^2$, $D_f = \mathbb{R}^2$ = alle tallpar (x,y) med x,y reelle tall



Eks: $f(x,y) = 2x+3y$, $D_f = \mathbb{R}^2$

$$\boxed{f(x,y) = \frac{x+y}{x-y}}, \quad D_f: x-y \neq 0$$



$$D_f = \{(x,y) : x-y \neq 0\}$$

Verdimengde: V_f = alle verdier (tall) z slik at $z = f(x,y)$ for et tall (x,y) i D_f.

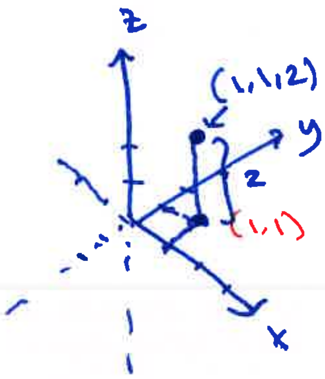
Eks:
$f(x,y) = 2x+3y$ $\quad V_f = \mathbb{R}$

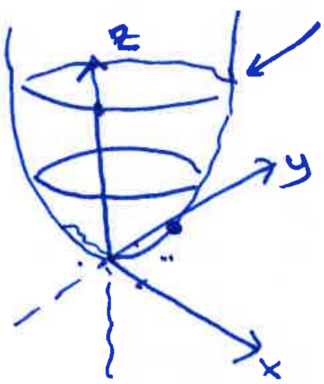
$f(x,y) = x^2+y^2$ $\quad V_f = [0, \to)$

$f(x,y) = \dfrac{x+y}{x-y}$

For å finne V_f, må vi finne max/min til f.

② **Grafen til f:** Grafen til f består av alle punkt (x,y,z) i $\mathbb{R}^3$ slik at $z = f(x,y)$ og (x,y) i D_f.





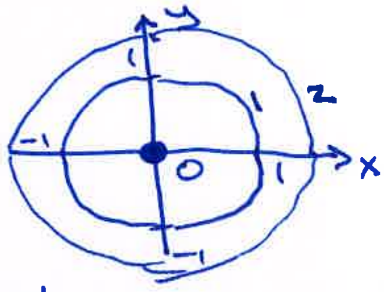
grafen til $f(x,y) = x^2+y^2$

Eks: $f(x,y) = x^2+y^2$

$(1,1)$: $z = f(1,1)$
$= 1^2+1^2 = 2$
$\Downarrow$
$(1,1,2)$

$D_f = \mathbb{R}^2$

Nivåkurver: Når vi setter $f(x,y) = c$ for en konstant c, får vi nivåkurven til f i høyde $z=c$. Dette er punktene på grafen til f i høyde $z=c$.



Eks: $f(x,y) = x^2+y^2$

Nivåkurven i høyde 1: $f(x,y)=1$

$x^2+y^2=1$

$\uparrow$

sirkel med senter i $(0,0)$ og radius $r=\sqrt{1}=1$

Nivåkurven i høyde 2: $f(x,y)=2$

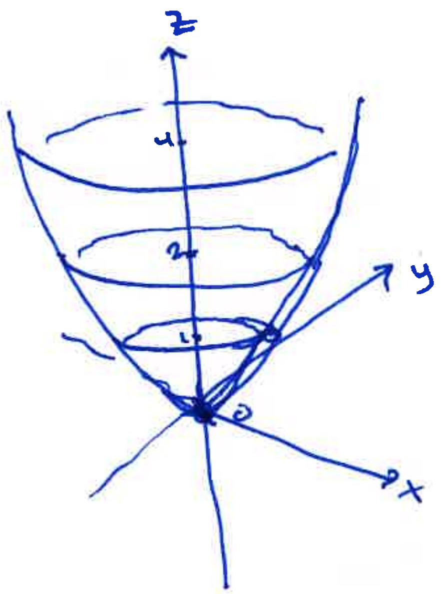
$x^2+y^2=2$

Sirkel m/ senter $(0,0)$, $r=\sqrt{2}$

Nivåkurven i høyde 0: $f(x,y)=0$

$x^2+y^2=0$

pkt. $(0,0)$



$f(x,y)=c$: sirkel m/senter $(0,0)$ og $r=\sqrt{c}$, $c>0$

pkt $(0,0)$, $c=0$

ingen pkt, $c<0$

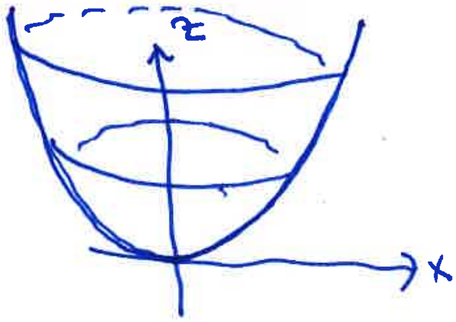
Vertikale kutt: Setter $x=a$ eller $y=b$

Eks: $f(x,y)=x^2+y^2$

$y=0$: $z=f(x,y)$ med $y=0$

$z=f(x,0)=x^2+0^2$

$z=x^2$



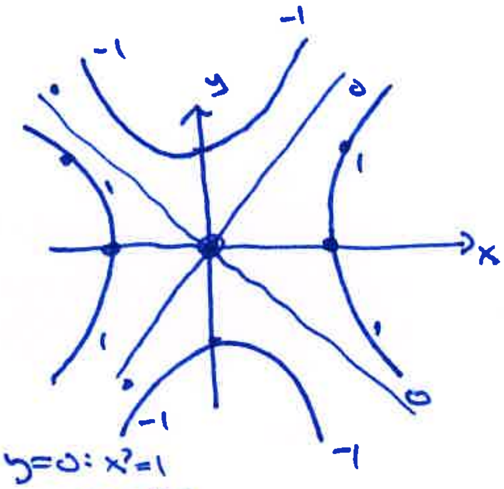
Eks: $f(x,y)=x^2-y^2$

Nivåkurver: $f=0$: $x^2-y^2=0$

$(x-y)(x+y)=0$

$x-y=0$ eller $x+y=0$

$y=x$ $\quad$ $y=-x$



$y=0$: $x^2=1$

$x=\pm 1$

$y=1$: $x^2=2$

$x=\pm\sqrt{2}$

$f=1$: $x^2-y^2=1$

$(x-y)(x+y)=1$

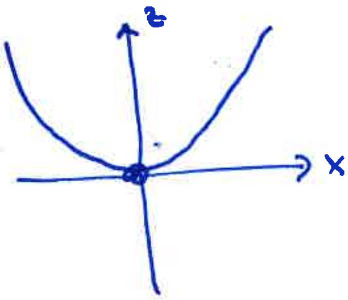
$x+y=\dfrac{1}{x-y}$

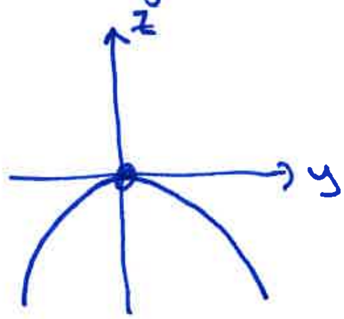
$f=-1$: $x^2-y^2=-1$

$y^2-x^2=1$

Kutt: $y=0$ $\qquad$ $x=0$

$z=x^2$ $\qquad$ $z=-y^2$





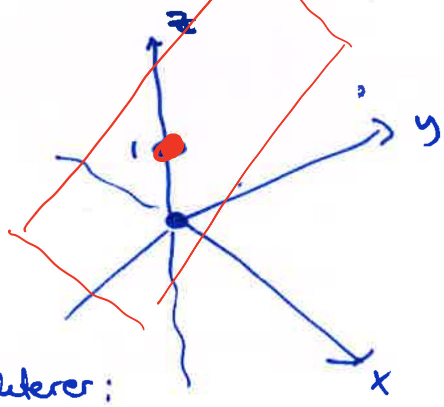
③ Lineære funksjoner

Resultat: $\boxed{f \text{ er en lineær funksjon}} \iff \boxed{\text{grafen til } f \text{ er et plan}}$

lineær funksjon:
$f(x,y) = ax+by+c$

plan: flate
uten krumning

Hvis $f(x,y) = ax+by+c$ er lineær, så er
c = skjæring med z-aksen: $f(0,0)=c$



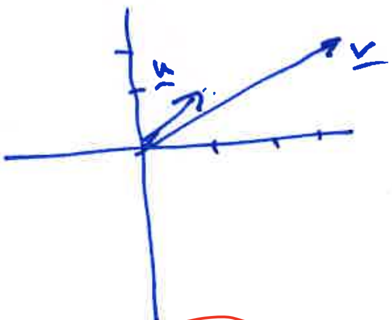
Eks: $f(x,y) = x-3y+1$
$f(0,0) = 0-3\cdot 0+1 = 1$

Indreprodukt (prikkprodukt) av vektorer:

$\left.\begin{array}{l} \underline{u} = (u_1,u_2,u_3) \\ \underline{v} = (v_1,v_2,v_3) \end{array}\right\}$ $\boxed{\underline{u}\cdot\underline{v} = u_1\cdot v_1 + u_2\cdot v_2 + u_3\cdot v_3}$

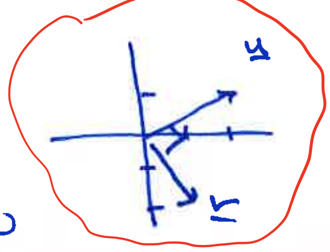
$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

Eks: $\underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\underline{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
$\underline{u}\cdot\underline{v} = 1\cdot 3 + 1\cdot 2$
$= 5$



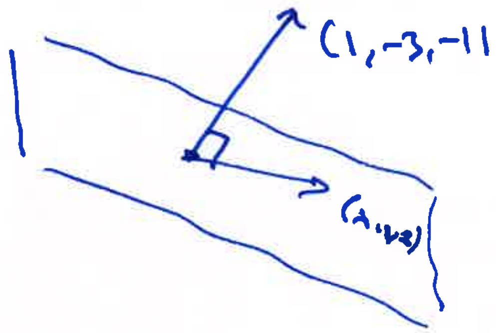
Resultat:
$\underline{u}\cdot\underline{v} = 0$
$\Updownarrow$
Vinkel mellom $\underline{u}$ og $\underline{v}$ er $90°$

Eks: $\underline{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
$\underline{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$
$\underline{u}\cdot\underline{v} = 2\cdot 1 + 1\cdot(-2) = 0$



Eks: $f(x,y) = x - 3y \Rightarrow$

$$z = x - 3y$$
$$0 = x - 3y - z$$
$$\underline{x - 3y - z = 0}$$



$$(1,-3,-1)\cdot(x,y,z) = 0$$

Oppsummering:

Forelesning 2c Del I (b):

$$\underline{u}\cdot\underline{v} = (u_1, u_2, u_3)\cdot(v_1, v_2, v_3)$$

$$= u_1\cdot v_1 + u_2\cdot v_2 + u_3\cdot v_3$$

eller

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

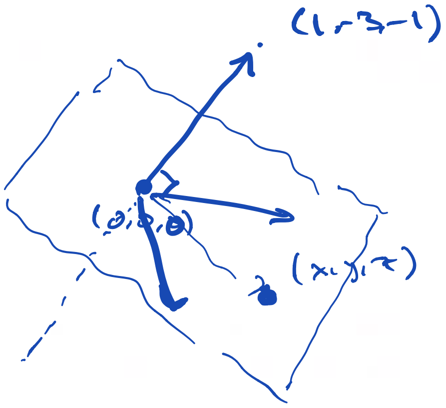
Resultat: $\underline{u}\cdot\underline{v} = 0 \iff \underline{u} \perp \underline{v}$ (vinkelen mellom $\underline{u}$ og $\underline{v}$ er $90°$, $\underline{u}$ står normalt på $\underline{v}$)

Eks: $f(x,y) = x - 3y$

$$\boxed{z = x - 3y} \iff x - 3y - z = 0$$

$$\boxed{(1,-3,-1)\cdot(x,y,z) = 0}$$

Vi kaller $(1,-3,-1)$ normalvektoren til planet



Alle vektor normalt på $(1,-3,-1)$:

$$(1,-3,-1)\cdot(x,y,z) = 0$$

$$\boxed{x - 3y - z = 0}$$

$$\left(1 \quad -3 \quad -1 \mid 0\right)$$

y, z frie var.

$$\underline{x} - 3y - z = 0$$

$$x = 3y + z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3y+z \\ y \\ z \end{pmatrix} = y\cdot\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Generelt:

$z = ax + by$

$0 = ax + by - z$

$f(x,y) = ax + by + \underline{c}$ lineær funksjon

Grafen er et plan som skjærer z-aksen i $\underline{z = c}$ og har normalvektor $(a, b, -1)$

Del 2:

Fagoppgaven, Oppg. 5 $f(x) = \dfrac{x^3-7x}{x^2-3x+2}$

a) Asymptoter:

i) Vertikale: $x^2-3x+2=0$

$x=2$, eller $x=1$

teller: $x=2: -6 \neq 0$

$x=1: -6 \neq 0$

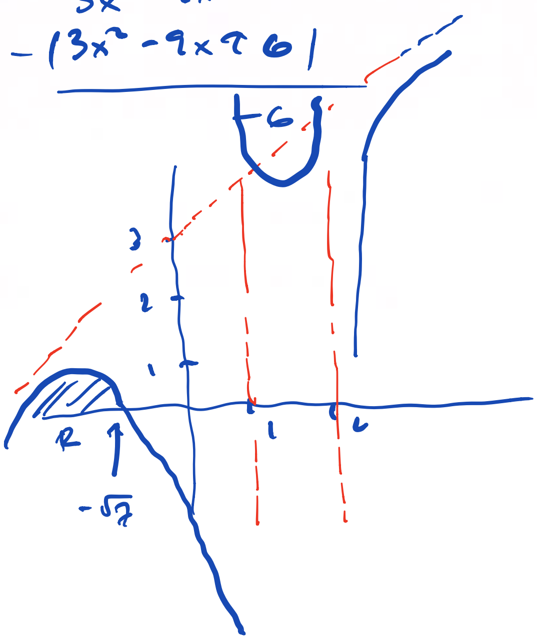
Konklusjon: $x=1$ og $x=2$ er vertikale asymptoter

ii) Horisontale / skrå.

$$(x^3-7x):(x^2-3x+2) = x+3$$
$$-(x^3-3x^2+2x)$$
$$3x^2-9x$$
$$-(3x^2-9x+6)$$
$$+6$$

$f(x) = x+3 + \dfrac{-6}{x^2-3x+2}$

Konklusjon: $y = x+3$ er skrå asymptote



$f(x) = \dfrac{x\cdot(x-\sqrt{7})(x+\sqrt{7})}{(x-1)(x-2)}$

Oppgaver 26

1.d) $f(x,y) = 17x^{1.2}y^{3.4}$, $D_f = \{(x,y) : x \geq 0, y \geq 0\}$

$V_f = [0, \to)$

$y^{3.4} = y^{\frac{34}{10}} = y^{\frac{17}{5}} = \sqrt[5]{y^{17}}$

$= \sqrt[10]{y^{34}}$

2.d) $\underline{v} = \begin{pmatrix} 4 \\ 7 \\ -3 \end{pmatrix}$: $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ står normalt på $\underline{v}$ $(\underline{x} \perp \underline{v})$

$\Updownarrow$

$\underline{v} \cdot \underline{x} = 0$ $\qquad \begin{pmatrix} 4 \\ 7 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$\Updownarrow$

$4x + 7y - 3z = 0$

$\frac{4x}{4} = \frac{-7y + 3z}{4}$

$$\boxed{x = -\frac{7}{4}y + \frac{3}{4}z \quad y, z \text{ frie}}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -(7/4)y + (3/4)z \\ y \\ z \end{pmatrix} \checkmark$$

$$= \frac{y}{4} \cdot \begin{pmatrix} -7 \\ 4 \\ 0 \end{pmatrix} + \frac{z}{4} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \qquad = y \cdot \begin{pmatrix} -7/4 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 3/4 \\ 0 \\ 1 \end{pmatrix}$$

3d $f(x,y) = x^2 - 2x + 4y^2$

$f(x,y) = c$ for $c = -2, -1, 0, 1, 2$

$x^2 - 2x + 4y^2 = c \quad | +1$

$x^2 - 2x + 1 + 4y^2 = c + 1$

$(x-1)^2 + 4y^2 = c+1$

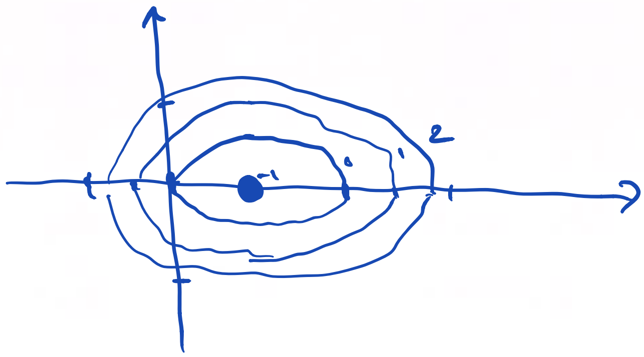
$$\frac{(x-1)^2}{c+1} + \frac{4 \cdot y^2}{c+1} = \frac{c+1}{c+1}$$

$$\frac{(x-1)^2}{c+1} + \frac{y^2}{(c+1)/4} = 1$$

$c > -1$: Ellipse m/senter $(1,0)$
Halvakser $a = \sqrt{c+1}$
$b = \frac{\sqrt{c+1}}{2}$

$c = -1$ punkt $(1,0)$

$c < -1$: ingen punkt.

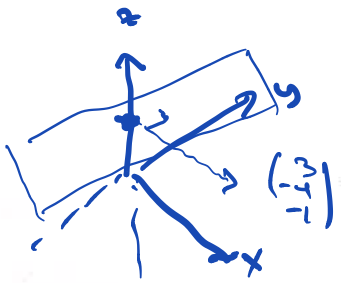


7.) Beskriv grafen til $f(x,y) = 3x - 4y + 1$ geometrisk:

Grafen er et plan, skjærer z-aksen i $z = 1$ med normalvektor $\begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}$

fordi $f(x,y)$ er lineær
" 1 er konstant-ledd. $f(x,y)$

fordi $z = 3x - 4y$
$3x - 4y - z = 0$
$\Updownarrow$
$\begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} \perp \begin{pmatrix} x \\ y \\ z \end{pmatrix}$



6. c) $f(x,y) = \dfrac{xy}{2x-y} = \dfrac{u}{v}$ $\quad u'_x = 1 \cdot y = y$, $\quad v'_x = 2$

x var.
y konst.

$\checkmark \; f'_x = \dfrac{u'_x v - u v'_x}{v^2} = \dfrac{(y \cdot 1) \cdot (2x-y) - xy \cdot (2)}{(2x-y)^2}$

$= \dfrac{\cancel{2xy} - y^2 - \cancel{2xy}}{(2x-y)^2} = \dfrac{-y^2}{(2x-y)^2}$

x konst.
y var.

$\checkmark \; f'_y = \dfrac{u'_y v - u v'_y}{v^2} = \dfrac{x \cdot (2x-y) - xy \cdot (-1)}{(2x-y)^2}$

$= \dfrac{2x^2 - \cancel{xy} + \cancel{xy}}{(2x-y)^2} = \dfrac{2x^2}{(2x-y)^2}$