Plan

1. Regneregler for matriser
2. Eksamensoppgaver 05/2018, opps 3

① Regneregler for matriser

a) Elementære radoperasjoner.

Bruker utvidet matrise $(A\,|\,\underline{b})$

b) Addisjon, subtraksjon, multiplikasjon

$A+B$, $A-B$: definert hvis A og B har samme størrelse

$c \cdot A$ (skalar multiplikasjon)

$\underset{m\times n}{A} \cdot \underset{n\times p}{B} \rightsquigarrow \underset{m\times p}{AB}$: definert hvis # kolonner i A = # rader i B

Regneregler:

i) $A\cdot B \neq B\cdot A$ matrisemultiplikasjon er ikke kommutativ

ii) $c\cdot A = A\cdot c$ når c er en skalar (et tall)

iii) andre regneregler for matriser er stort sett som for vanlige tall (vanlig algebra)

For eksempel:
$$A\cdot(B+C) = A\cdot B + A\cdot C$$
$$A+B = B+A$$
$$A\cdot(BC) = (AB)C$$

c) Determinant: $|A|$ definert når A er kvadratisk $(n \times n)$

Regneregler:
i) $|AB| = |A| \cdot |B|$ ✓
ii) $|c \cdot A| = c^n \cdot |A|$ når A er $n \times n$
iii) $|A^T| = |A|$
iv) $|A^{-1}| = \frac{1}{|A|}$

$A \cdot A^{-1} = I$
$|A \cdot A^{-1}| = |I|$
$|A| \cdot |A^{-1}| = 1$

Husk: Hvis A er øvre triangulær

$$A = \begin{pmatrix} a_{11} & & * \\ & a_{22} & \\ 0 & & \ddots \; a_{nn} \end{pmatrix} \Rightarrow |A| = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$$

Eksempel:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

$$|A| \cdot |B| = (ad - bc) \cdot (\cancel{xz - yz}\ xw - yz) \checkmark$$

$$|AB| = \begin{vmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{vmatrix} = (ax+bz)(cy+dw) - (ay+bw)(cx+dz) \checkmark$$

d) Transponert: $\underset{m \times n}{A} \longrightarrow \underset{n \times m}{A^T}$

Regneregler:
i) $(A \pm B)^T = A^T \pm B^T$
ii) $(cA)^T = c \cdot A^T$ c skalar
iii) $(AB)^T = B^T \cdot A^T$ ✓
iv) $(A^T)^T = A$

Defn: A kalles en symmetrisk matrise hvis $A^T = A$.

Eks: $$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 7 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 7 \end{pmatrix}$$

e) Invers: $\boxed{\begin{matrix} A \cdot A^{-1} = I \\ A^{-1} \cdot A = I \end{matrix}}$ A^{-1} fins $\Leftrightarrow$ $\begin{cases} A \text{ er kvadratisk} \\ \text{og} \\ |A| \neq 0 \end{cases}$

Regneregler:
i) $(A^{-1})^{-1} = A$
ii) $(AB)^{-1} = B^{-1}A^{-1}$ ✓

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{12} & \ldots \\ \vdots & & \\ C_{n1} & C_{n2} & \end{pmatrix}^T$$

Merk: ① $(AB)^{-1} = B^{-1}A^{-1}$ ✓

$(AB)\cdot B^{-1}A^{-1}$
$= A\cancel{B}\cancel{B^{-1}}A^{-1} = \cancel{A}\cancel{A^{-1}} = I$
$\cancel{B^{-1}}\cancel{A^{-1}}\cdot(\cancel{A}\cancel{B}) = I$

② $A\underline{x} = A\cdot\underline{y}$

$|A| \neq 0$ $\qquad$ $|A| = 0$

?

$\cancel{A^{-1}}\cdot\cancel{A}\underline{x} = \cancel{A^{-1}}\cdot\cancel{A}\underline{y}$

$\underline{x} = \underline{y}$

② Eksamen MET11803 05/2018 Oppgave 3

$$A = \begin{pmatrix} 2 & a & 0 \\ a & 1 & a \\ 0 & a & 2 \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 3 \\ a \\ -a \end{pmatrix}$$

Vi ser på $A\cdot\underline{x} = \underline{b}$ der a er parameter.

a) Løs systemet for $a=1$:

$\underline{a=1}$:
$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) \begin{matrix}]-1 \\ \\ \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) \begin{matrix}]-1 \\ \\ \end{matrix}$$

utvidet matrise

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \end{array}\right) \begin{matrix} \\ ]-1 \\ \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

trappeform
$\underline{z \text{ fri}}$

$\underline{x} \quad -z = 2 \qquad x = \underline{z+2}$
$y + 2z = -1 \qquad y = \underline{-2z-1}$

$\Rightarrow (x,y,z) = \underline{\underline{(z+2, -2z-1, z)}}$ der z er fri

b) $|A| = \begin{vmatrix} 2 & a & 0 \\ a & 1 & a \\ 0 & a & 2 \end{vmatrix} = 2\cdot(2-a^2) - a(2a) + 0$

$= 4 - 2a^2 - 2a^2 = \underline{\underline{4-4a^2}} = 4(1-a^2) = \underline{\underline{4(1-a)(1+a)}}$

$\underline{|A|=0}$: $4(1-a)(1+a) = 0 \Rightarrow \underline{\underline{a=1}}$ eller $\underline{\underline{a=-1}}$

c) $\underline{\text{Bestem } a}$ slik at $A\underline{x}=\underline{b}$ har uendelig mange løsn.

$|A| \neq 0$: $a \neq \pm 1 \Rightarrow$ én løsning

$|A| = 0$: $a = 1$: $\underline{\text{Uendelig mange løsninger}}$ fra (a).

$a=-1$: $\underline{\text{Ingen løsn.}}$

$\underline{a=-1}$:

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 3 \\ -1 & 1 & -1 & -1 \\ 0 & -1 & 2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ -1 & 1 & -1 & -1 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

utvidet matrise

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

trappeform

$\underline{\text{Ingen løsn.}}$

$\underline{\text{Konklusjon}}$: Uendelig mange løsninger for $\underline{\underline{a=1}}$

d) Regn ut $\underline{x}^T A \underline{x}$ når $a=1$

$a=1$: $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\underline{x}^T = (x\ y\ z)$

$$(\underline{x}^T A)\cdot \underline{x} = \underbrace{(x\ y\ z)\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}}\cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\underline{x}^T \cdot (A\cdot x)$

$$= \begin{pmatrix} 2x+y & x+y+z & y+2z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \left((2x+y)x + (x+y+z)\cdot y + (y+2z)\cdot z \right)$$

$$= \left(2x^2 + xy + xy + y^2 + yz + yz + 2z^2 \right)$$

$$= \left(\underline{\underline{2x^2 + 2xy + y^2 + 2yz + 2z^2}} \right)$$

Alternative spørsmål:

e) Finn løsningen av $A\underline{x} = \underline{b}$ i de tilfellene der det er én løsning.

To metoder: – Kramers regel

– Bruke A^{-1}: $A\underline{x} = \underline{b} \Rightarrow \underline{x} = A^{-1}\cdot \underline{b}$

(– Gauss)

i) Kramer's regel: $a \neq \pm 1$ $\quad |A| = 4(1-a)(1+a) \neq 0$

$$A = \begin{pmatrix} 2 & a & 0 \\ a & 1 & a \\ 0 & a & 2 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 3 \\ a \\ -a \end{pmatrix}$$

$$x = \frac{|A_x(\underline{b})|}{|A|} = \frac{\begin{vmatrix} 3 & a & 0 \\ a & 1 & a \\ -a & a & 2 \end{vmatrix}}{4(1-a)(1+a)} = \frac{3(2-a^2) - a(2a+a^2)}{4(1-a)(1+a)}$$

$$= \frac{6 - 5a^2 - a^3}{4(1-a)(1+a)}$$

$$y = \frac{|A_y(\underline{b})|}{|A|} = \frac{\begin{vmatrix} 2 & 3 & 0 \\ a & a & a \\ 0 & -a & 2 \end{vmatrix}}{4(1-a)(1+a)} = \frac{2(2a+a^2) - 3(2a)}{4(1-a)(1+a)}$$

$$= \frac{-2a + 2a^2}{4(1-a)(1+a)} = \frac{\cancel{2}a\cancel{(-1+a)}^{\,-1}}{\cancel{2} \cdot 2\cancel{(1-a)}(1+a)} = \frac{-1}{2(1+a)}$$

$$z = \frac{|A_z(\underline{b})|}{|A|} = \frac{\begin{vmatrix} 2 & a & 3 \\ a & 1 & a \\ 0 & a & -a \end{vmatrix}}{4(1-a)(1+a)} = \frac{2(-a-a^2) - a(-a^2-3a)}{4(1-a)(1+a)}$$

$$= \frac{-2a + a^2 + a^3}{4(1-a)(1+a)}$$

ii) Invers matrise: $|A| = 4(1-a)(1+a) \neq 0$ når $a \neq \pm 1$

$$A^{-1} = \frac{1}{4(1-a)(1+a)} \cdot \begin{pmatrix} 2-a^2 & -2a & a^2 \\ -2a & 4 & -2a \\ a^2 & -2a & 2-a^2 \end{pmatrix}^T \qquad A = \begin{pmatrix} 2 & a & 0 \\ a & 1 & a \\ 0 & a & 2 \end{pmatrix}$$

symmetrisk

Merk: A er symmetrisk $\Rightarrow$ C symmetrisk

$$A^{-1} = \frac{1}{4(1-a)(1+a)} \cdot \begin{pmatrix} 2-a^2 & -2a & a^2 \\ -2a & 4 & -2a \\ a^2 & -2a & 2-a^2 \end{pmatrix}$$

$$A\underline{x} = \underline{b} \Rightarrow \underline{x} = A^{-1} \cdot \underline{b} = \frac{1}{4(1-a)(1+a)} \cdot \begin{pmatrix} 2-a^2 & -2a & a^2 \\ -2a & 4 & -2a \\ a^2 & -2a & 2-a^2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ a \\ -a \end{pmatrix}$$

$$= \frac{1}{4(1-a)(1+a)} \cdot \begin{pmatrix} 3(2-a^2) - 2a^2 - a^3 \\ -6a + 4a + 2a^2 \\ 3a^2 - 2a^2 - a(2-a^2) \end{pmatrix}$$

$$= \frac{1}{4(1-a)(1+a)} \cdot \begin{pmatrix} 6 - 5a^2 - a^3 \\ -2a + 2a^2 \\ -2a + a^2 + a^3 \end{pmatrix}$$

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2c)

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 3 & 1 \end{pmatrix}$$

$$A^T \cdot A = \underbrace{\begin{pmatrix} 1 & 3 \\ -1 & 3 \\ 3 & 1 \end{pmatrix}}_{A^T} \cdot \underbrace{\begin{pmatrix} 1 & -1 & 3 \\ 3 & 3 & 1 \end{pmatrix}}_{A} = \begin{pmatrix} 10 & 8 & 6 \\ 8 & 10 & 0 \\ 6 & 0 & 10 \end{pmatrix}$$

3. d)

$$A = \begin{pmatrix} a & 1 & a \\ 1 & 2 & 3 \\ a & 3 & 0 \end{pmatrix} \quad \to a = 1: \; A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 0 \end{pmatrix}$$

$$A^2 - 3A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 0 \end{pmatrix} - 3 \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 6 & 4 \\ 6 & 14 & 7 \\ 4 & 7 & 10 \end{pmatrix} - \begin{pmatrix} 3 & 3 & 3 \\ 3 & 6 & 9 \\ 3 & 9 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 1 \\ 3 & 8 & -2 \\ 1 & -2 & 10 \end{pmatrix}$$

4.

$$A = \begin{pmatrix} 2-s & 3 & 3 \\ 3 & 2-s & 3 \\ 3 & 3 & 2-s \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 3 \\ s+4 \\ 1-2s \end{pmatrix}$$

$A\underline{x} = \underline{b}$ (parameter s)

a) $s = 8$:

$$\left(\begin{array}{ccc|c} -6 & 3 & 3 & 3 \\ 3 & -6 & 3 & 12 \\ 3 & 3 & -6 & -15 \end{array}\right) \to \left(\begin{array}{ccc|c} -3 & -3 & 6 & 15 \\ 3 & -6 & 3 & 12 \\ 3 & 3 & -6 & -15 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} -3 & -3 & 6 & 15 \\ 0 & -9 & 9 & 27 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-3x - 3y + 6z = 15$

$-9y + 9z = 27$

$-3x = 15 + 3y - 6z$

$x = -5 - (z-3) + 2z = z - 2$

$\frac{-9y}{-9} = \frac{27 - 9z}{-9}$

$y = z - 3$

trappeform

z fri $\Rightarrow$ én frihetsgrad (antall frie var.)

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Løsning: $(x, y, z) = (z-2, z-3, z)$ med z fri

b) $|A| = \begin{vmatrix} 2-s & 3 & 3 \\ 3 & 2-s & 3 \\ 3 & 3 & 2-s \end{vmatrix} = \underline{(2-s)\cdot\left[(2-s)^2 - 9\right]} - 3(3(2-s)-9)$
$+3(9-3(2-s))$

$s^2-4s-5=0$

$s = \frac{4 \pm \sqrt{4^2 - 4\cdot(-5)}}{2}$

$= 2 \pm 3$

$= 5, -1$

$= \underbrace{(2-s)(s^2-4s-5)}_{(2-s)\,(s-5)(s+1)} + \underbrace{9+9s+9+9s}_{18s+18 = 18(s+1)}$

$= (s+1)\cdot\left((2-s)(s-5)+18\right)$

$= (s+1)\cdot(-s^2+7s+8) = -(s+1)(s^2-7s-8)$

$= -(s+1)(s+1)(s-8) = \underline{\underline{-(s+1)^2(s-8)}}$

c) $\underline{s=0}$: $A = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$ $|A| = -1^2\cdot(-8) = \underline{8} \neq 0$

$A^{-1} = \frac{1}{8}\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{8}\begin{pmatrix} -5 & 3 & 3 \\ 3 & -5 & 3 \\ 3 & 3 & -5 \end{pmatrix}^T = \underline{\underline{\frac{1}{8}\begin{pmatrix} -5 & 3 & 3 \\ 3 & -5 & 3 \\ 3 & 3 & -5 \end{pmatrix}}}$

$A^{-1}\cdot|$ $\quad A\underline{x} = \underline{b} \Rightarrow \underline{x} = A^{-1}\cdot\underline{b} = \frac{1}{8}\begin{pmatrix} -5 & 3 & 3 \\ 3 & -5 & 3 \\ 3 & 3 & -5 \end{pmatrix}\cdot\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$

$\underline{x} = \frac{1}{8}\begin{pmatrix} 0 \\ -8 \\ 16 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}}}$

d) Én løsning av $A\underline{x} = \underline{b} \Leftrightarrow |A| \neq 0$

$|A| = -(s+1)^2(s-8) = 0 \Leftrightarrow s=-1, s=8$

Én løsning for $\underline{\underline{s \neq -1, 8}}$

Kramers regel:

$s \neq -1, 8$: $\quad x = \frac{|A_x(\underline{b})|}{|A|} = \frac{\begin{vmatrix} 3 & 3 & 3 \\ s+4 & 2-s & 3 \\ 1-2s & 3 & 2-s \end{vmatrix}}{-(s+1)^2(s-8)} = \frac{0}{|A|} = \underline{\underline{0}}$

$$\begin{vmatrix} 3 & 3 & 3 \\ s+4 & 2-s & 3 \\ 1-2s & 3 & 2-s \end{vmatrix} = 3\left((2-s)^2-9\right) - 3\left((s+4)(2-s) - 3(1-2s)\right) + 3\left(3(s+4) - (2-s)(1-2s)\right)$$

$$= 3(s^2-4s-5) - 3(-s^2-2s+8-3+6s) + 3(3s+12-2s^2+5s-2)$$

$$= \cancel{3s^2} - \cancel{12s} - \cancel{15} + \cancel{3s^2} + \cancel{6s} - \cancel{15} - \cancel{18s} - \cancel{6s^2} + \cancel{24s} + \cancel{30} = 0$$