Plan

1 Matrisemultiplikasjon
2 Regning med matriser
3 Inverse matriser

① Matrisemultiplikasjon:

Husk: Addisjon / subtraksjon : $A+B$, $A-B$

Eks: $\begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 0 & 7 \end{pmatrix}$

Skalarmultiplikasjon: $r \cdot A$

Eks: $3 \cdot \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 0 & 12 \end{pmatrix}$

Defn: Matrisemultiplikasjon: $A \cdot B$

- $A \cdot B$ er definert hvis # kolonner i A = # rader i B

$$\boxed{\underset{m \times n}{A} \quad \underset{n \times p}{B} \longrightarrow \underset{m \times p}{A \cdot B}}$$

- I så fall vil $A \cdot B$ ha like mange rader som A og like mange kolonner som B
- Vi får $AB = (c_{ij})$

der $c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \ldots + a_{in} \cdot b_{nj}$

Eks: $\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \end{pmatrix}$

$2 \cdot 1 + 3 \cdot 2 = 8$

$-1 \cdot 1 + 4 \cdot 2 = 7$

$2 \times \underline{2} \quad \underline{2} \times 1$

$=$

Eks:

$$\begin{pmatrix} 1 & 4 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ -2 & 1 \end{pmatrix}$$

$$2 \times 2 = 2 \times 2$$

$$\begin{array}{c|c} & \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = B \\ \hline A = \begin{pmatrix} 1 & 4 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 6 & 3 \\ -2 & 1 \end{pmatrix} = AB \end{array}$$

Matrisemultiplikasjon og lineære system

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

koeff. matrisen

$$A \cdot \underline{x} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}}_{m \times n} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{n \times 1} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix}$$

$$= \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \underline{b}$$

Lineært likningssystem på matriseform:

$$\boxed{A \cdot \underline{x} = \underline{b}}$$

(2) **Kommentar:** Når A er kvadratisk $n \times n$-matrise, så kan vi finne A^m

potenser av kvadratiske matriser

$$\begin{cases} A^2 = A \cdot A & n \times n \\ A^3 = (A \cdot A) A = A^2 \cdot A & n \times n \\ \vdots \\ A^m = A^{m-1} \cdot A \quad \text{for } m \geq 2 \end{cases}$$

Defn: Den transponerte matrisen til A = A^T

$$\underset{m \times n\text{-matrise}}{A} \longrightarrow \underset{n \times m\text{-matrise}}{A^T}$$

"bytter om rader og kolonner"

Eks: $A = \begin{pmatrix} 7 & 1 \\ 2 & -1 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} 7 & 2 \\ 1 & -1 \end{pmatrix}$

Kommentar: $A \cdot B \neq B \cdot A$ for matrisemultiplikasjon

3 Inverse matriser

Defn: Identitetsmatrisen I er den kvadratiske matrise

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For tall:

divisjon med a = multiplikasjon med $\frac{1}{a}$ (hvis $a \neq 0$)

$\frac{1}{a} = a^{-1}$ er den inverse til tallet a

Egenskap: $A \cdot I = A$ for enhver matrise A
$I \cdot A = A$

$$2x = 6$$
$$\tfrac{1}{\cancel{2}} \cdot \cancel{2}x = \tfrac{1}{2} \cdot 6$$
$$x = 3$$

Eks:

$$\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$$

Defn: La A være en matrise. Hvis det fins en matrise B slik at

$$A \cdot B = I$$
$$B \cdot A = I$$

så kalles B en invers matrise til A.

Resultat: i) Hvis A har en invers matrise, så er den entydig, og den skrives A^{-1}. $(B = A^{-1})$

ii) En matrise A har en invers hvis og bare hvis A er kvadratisk og $|A| \neq 0$.

Eks: $A = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ $\quad |A| = 1\cdot 2 - 3\cdot(-1) = 5 \neq 0$

$\Downarrow$

A har en entydig invers

$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$a + 3c = 1$
$b + 3d = 0$
$-a + 2c = 0$
$\vdots$

Formel for 2×2-matriser:

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$:

i) Hvis $|A| = a_{11}a_{22} - a_{12}a_{21} \neq 0$, så er

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \cdot \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

ii) Hvis $|A| = 0$, så er det ingen invers.

Eks: $A = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ $\quad |A| = 5 \neq 0 \Rightarrow A^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 2/5 & -3/5 \\ 1/5 & 1/5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2/5 & -3/5 \\ 1/5 & 1/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{1}{5}\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Anvendelse

$$\left\{ \begin{matrix} x + 3y = 14 \\ -x + 2y = 6 \end{matrix} \right\} \qquad \underbrace{\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 6 \end{pmatrix}}_{\text{matriseform}}$$

$A\underline{x} = \underline{b}$

$A^{-1} \cdot A\underline{x} = A^{-1} \cdot \underline{b}$

$\underline{x} = I \cdot \underline{x} = A^{-1}\underline{b}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \cdot \underline{b} = \underbrace{\frac{1}{5}\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}}_{A^{-1}} \cdot \underbrace{\begin{pmatrix} 14 \\ 6 \end{pmatrix}}_{\underline{b}} = \frac{1}{5} \cdot \begin{pmatrix} 10 \\ 20 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 \\ 4 \end{pmatrix}}}$$

<u>Eks:</u> $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\quad |A| = 4 \cdot 2 \cdot 2 = \underline{0} \Rightarrow A^{-1}$ <u>finnes ikke</u>

<u>Generelt tilfelle</u>: A $n \times n$-matrise

$|A| \neq 0$: $\quad A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A) = \frac{1}{|A|} \cdot C^T$

den adjungerte matrisen til A

$\underline{|A| = 0}$: $\quad A^{-1}$ <u>finnes ikke</u>

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}$$

kofaktormatrise

Del 2:

Repetisjon: – matrisemultiplikasjon
– inverse matriser (A^{-1} fins $\Leftrightarrow |A| \neq 0$)

Regneregler:
$$\boxed{\begin{aligned} |A\cdot B| &= |A|\cdot|B| \\ (A\cdot B)^T &= B^T \cdot A^T \end{aligned}}$$

Oppgavesett 24:

1. a)

$$C^T \cdot A = \underbrace{\overset{2\times 3}{\begin{pmatrix} 3 & 1 & 7 \\ 4 & -2 & 1 \end{pmatrix}}}_{C^T} \cdot \overset{3\times 3}{\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{pmatrix}}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & 4 \\ 1 & -2 \\ 7 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 11 & 14 \\ -1 & 3 & -3 \end{pmatrix}$$

2. d)

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \qquad |A| = 2\cdot 1\cdot 1 = \underline{2} \neq 0$$

$$C_{11} = +1 \quad C_{12} = -0 \quad C_{13} = +0$$
$$C_{21} = -1 \quad C_{22} = +2 \quad C_{23} = -0$$
$$C_{31} = -2 \quad C_{32} = -4 \quad C_{33} = +2$$

$$A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A) = \frac{1}{2} C^T$$

$$= \frac{1}{2}\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & -4 & 2 \end{pmatrix}^T = \underline{\underline{\frac{1}{2}\begin{pmatrix} 1 & -1 & -2 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{pmatrix}}}$$

3c.

$$A = \begin{pmatrix} 1 & 1 & a \\ 1 & 3 & 1 \\ a & 1 & 1 \end{pmatrix}$$

$$|A| = 1 \cdot 2 - 1 \cdot (1-a) + a \cdot (1-3a)$$
$$= 2 - 1 + a + a - 3a^2$$
$$= -3a^2 + 2a + 1$$

For $a \neq 1, -1/3$ finn A^{-1}

$|A| = 0$: $a = \dfrac{-2 \pm \sqrt{2^2 - 4(-3)\cdot 1}}{2 \cdot (-3)} = \dfrac{-2 \pm 4}{-6} = 1, -1/3$

$$A^{-1} = \frac{1}{-3a^2+2a+1} \cdot \begin{pmatrix} +2 & -(1-a) & 1-3a \\ -(1-a) & 1-a^2 & -(1-a) \\ 1-3a & -(1-a) & +2 \end{pmatrix}^T$$

$$A^{-1} = \frac{1}{1+2a-3a^2} \begin{pmatrix} 2 & a-1 & 1-3a \\ a-1 & 1-a^2 & a-1 \\ 1-3a & a-1 & 2 \end{pmatrix}, \quad a \neq 1, -1/3$$

4.

$$A = \begin{pmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \underline{b} = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}$$

Matrise-form: $\boxed{A \cdot \underline{x} = \underline{b}}$

a) $t = 2$:

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 2 & 2 \end{array}\right) \begin{matrix} -1 \\ \\ \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 2 & 2 \end{array}\right) \begin{matrix} \\ \\ -1 \end{matrix}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 2 \end{array}\right)$$

$x - z = 0 \quad x = z = 2/3$

$2y = 0 \quad y = 0$

$3z = 2 \quad z = 2/3$

Løsning: $(x, y, z) = (2/3, 0, 2/3)$

b) $$|A| = \begin{vmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{vmatrix} = t(t^2-1) = \underline{t(t-1)(t+1)}$$

$|A| = 0$: $\underline{t=0}$, $\underline{t=1}$, $\underline{t=-1}$

$\underline{t \neq 0, 1, -1}$: én løsning

$\underline{t=0}$: $$\begin{pmatrix} 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 1 & 0 & 0 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$ uendelig mange løsninger

$\underline{t=1}$: $$\begin{pmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 0 & | & 0 \\ 1 & 0 & 1 & | & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$ — ,, —

$\underline{t=-1}$: $$\begin{pmatrix} -1 & 0 & 1 & | & -1 \\ 0 & 1 & 0 & | & 0 \\ 1 & 0 & -1 & | & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & | & -1 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & -2 \end{pmatrix}$$ ingen løsn.

c) $|A| = t(t+1)(t-1)$

$|A| \neq 0 \Leftrightarrow t \neq 0, 1, -1$ gir at $\underline{A^{-1} \text{ fins}}$

$$A = \begin{pmatrix} \boxed{t} & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{pmatrix}$$

$\underline{t \neq 0, \pm 1}$: $$A^{-1} = \frac{1}{t(t^2-1)} \cdot \begin{pmatrix} t^2 & 0 & -t \\ 0 & t^2-1 & 0 \\ -t & 0 & t^2 \end{pmatrix}^T$$

$$= \frac{1}{t(t^2-1)} \cdot \begin{pmatrix} t^2 & 0 & -t \\ 0 & t^2-1 & 0 \\ -t & 0 & t^2 \end{pmatrix}$$

$A \cdot \underline{x} = \underline{b}$

$A^{-1} \cdot A\underline{x} = A^{-1}\underline{b}$

$\underline{x} = A^{-1}\underline{b}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\frac{1}{t(t^2-1)} \cdot \begin{pmatrix} t^2 & 0 & -t \\ 0 & t^2-1 & 0 \\ -t & 0 & t^2 \end{pmatrix}}_{A^{-1}} \cdot \underbrace{\begin{pmatrix} t \\ 0 \\ t \end{pmatrix}}_{\underline{b}}$$

$$= \frac{1}{t(t^2-1)} \cdot \begin{pmatrix} t^3-t^2 \\ 0 \\ t^3-t^2 \end{pmatrix} = \frac{1}{t(t-1)(t+1)} \begin{pmatrix} t^2(t-1) \\ 0 \\ t^2(t-1) \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{t}{t+1} \\ 0 \\ \frac{t}{t+1} \end{pmatrix} \quad \text{for } t \neq 0, 1, -1$$

5. b) $(A^T A)^T = A^T \cdot (A^T)^T = \underline{A^T \cdot A}$ $\qquad (AB)^T = B^T \cdot A^T$

e) $(BAB^{-1})^2 \cdot B^2 = (BA\cancel{B^{-1}})(\cancel{B}AB^{-1})\cancel{B} \cdot B$

$= BA \cdot A \cdot B = \underline{BA^2B}$

f) $(A-B)(C-A) + (C-B)(A-C) + (C-A)^2$

$= AC - BC - A^2 + \cancel{BA} + CA - \cancel{BA} - C^2 + BC + (C-A)\cdot(C-A)$

$= \cancel{AC} - BC - \cancel{A^2} + \cancel{CA} - \cancel{C^2} + B^2 + \cancel{C^2} - \cancel{AC} - \cancel{CA} + \cancel{A^2}$

$= \underline{BC - BC} = \underline{\underline{0}}$

6. A, B 3×3 med $|A| = 2$ $|B| = -5$

a) $|AB| = |A| \cdot |B| = 2 \cdot (-5) = \underline{-10}$

b) $|3A| = 3^3 \cdot |A| = 27 \cdot 2 = \underline{\underline{54}}$

$$\begin{array}{l} 3\cdot \\ 3\cdot \\ 3\cdot \end{array}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

c) $|-2B^T| = (-2)^3 \cdot |B^T| = -8 \cdot |B|$

$= (-8) \cdot (-5) = \underline{\underline{40}}$

d) $|2A^{-1} \cdot B| = 2^3 \cdot |A^{-1}| \cdot |B| = 8 \cdot |A^{-1}| \cdot (-5)$

$= 8 \cdot \frac{1}{2} \cdot (-5)$

$= \underline{\underline{-20}}$

$A \cdot A^{-1} = I$

$|A \cdot A^{-1}| = |I| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot 1 \cdot \ldots \cdot 1 = 1$

$|A| \cdot |A^{-1}| = 1 \quad \Rightarrow \quad \boxed{|A^{-1}| = 1/|A|}$