Plan
1 Økonomiske anvendelser av integrasjon
2 Partiellderivasjon

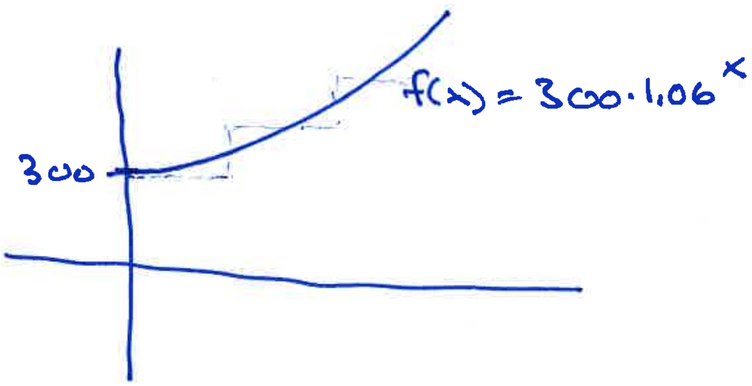
① Økonomiske anvendelser av integrasjon

(*) Kontinuerlige kontantstrømmer

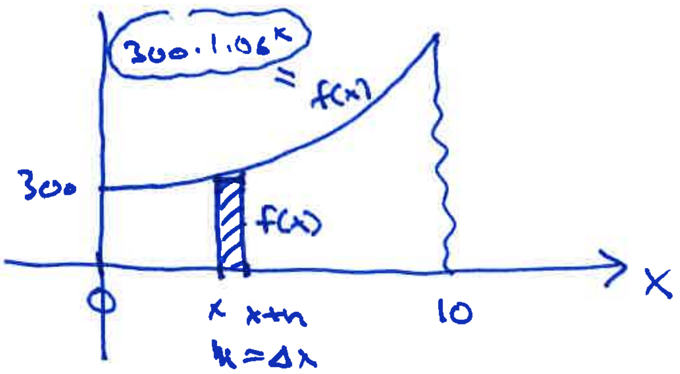
Eks: Anta at leien er gitt ved den kontinuerlige funksjonen

$$f(x) = 300 \cdot 1.06^x$$

x: tiden (år) $f(x)$ = leie (mill. kr / år)



Samlet leieinntekt i løpet av 10 år:



Samlet leieinntekt:

a^x

$$\int_0^{10} f(x)\,dx = \int_0^{10} 300 \cdot 1.06^x\,dx$$

$$= \left[300 \cdot a^x \cdot \frac{1}{\ln a}\right]_0^{10}$$

$$= \left[\frac{300}{\ln 1.06} \cdot 1.06^x\right]_0^{10}$$

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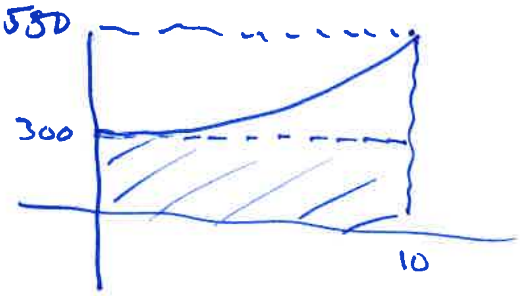
$$\frac{300}{\ln 1.06}\cdot 1.06^{10} - \frac{300}{\ln 1.06}\cdot 1.06^{0} = \frac{300}{\ln 1.06}\left(1.06^{10}-1\right)$$

$$\approx \underline{4.072} \quad (\text{mill kr})$$



Rule of 72:

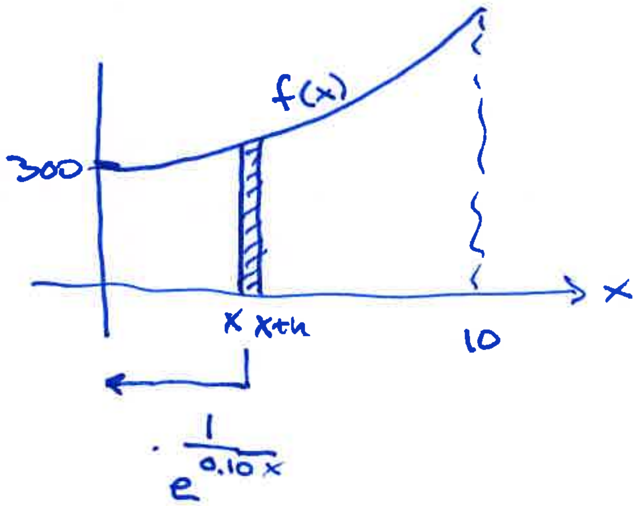
øker med 6% i år

$\Rightarrow$ doblingstid ca $\frac{72}{6} = 12$

Hva er nåverdien av denne kontantstrømmen:

Vi bruker kontinuerlig diskontering: $\cdot\, e^{-rt} = \cdot\, \frac{1}{e^{rt}}$

Anta $r = 10\%$ (diskonteringsrente)



Nåverdi:

$$\int_0^{10} f(x)\cdot e^{-rx}\cdot dx$$

$$= \int_0^{10} 300\cdot 1.06^{x}\cdot e^{-0.10x}\,dx$$

$$= \int_0^{10} 300\, e^{\ln(1.06)\cdot x}\cdot e^{-0.10x}\,dx$$

$$= \int_0^{10} 300\, e^{\ln(1.06)x - 0.10x}\,dx$$

$1.06^x = \left(e^{\ln 1.06}\right)^x$

$= e^{\ln(1.06)\cdot x}$

$$= \int 300\, e^{u}\cdot \frac{du}{\ln 1.06 - 0.10}$$

$$\boxed{\begin{aligned} u &= \ln(1.06)x - 0.10x \\ du &= (\ln 1.06 - 0.10)\,dx \end{aligned}}$$

$$= \left[\frac{300}{\ln 1.06 - 0.10}\, e^{u}\right]$$

$$= \left[\frac{300}{\ln 1.06 - 0.10}\, e^{\ln(1.06)x - 0.10x}\right]_0^{10}$$

$$= \frac{300}{\ln(1.06)-0.10}\, e^{10\cdot \ln 1.06 - 1} - \frac{300}{\ln 1.06 - 0.10}\, e^{0}$$

$$= \underbrace{\frac{300}{\ln 1.06 - 0.10}}_{-7189} \Big(\underbrace{\frac{1.06^{10}}{e} - 1}_{\simeq -0.34} \Big) \simeq 2.4527 \text{ mill kr}$$

Formler:

$f(x)$: kontantstrøm per tidsenhet (kont. fn.)

$[t_1, t_2]$: tidsintervallet vi ser på

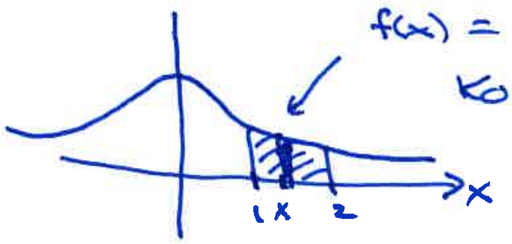
r : diskonteringsrente (kontinuerlig disk.)

$$\boxed{\begin{array}{ll} \text{Samlet kontantstrøm:} & \int_{t_1}^{t_2} f(x)\,dx \\ \text{Nåverdi av kontantstrøm:} & \int_{t_0}^{t_1} f(x)\, e^{-rx}\,dx \end{array}}$$

(B) Sannsynligheter med kontinuerlig tetthetsfunksjon

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

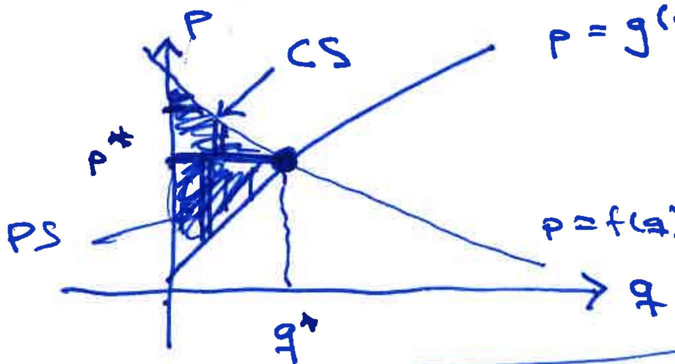
Kont. tetthetsfn.



$$P(1 \le X \le 2) = \int_1^2 f(x)\,dx$$

standard normalfordeling

(c) Konsument / produsent overskudd

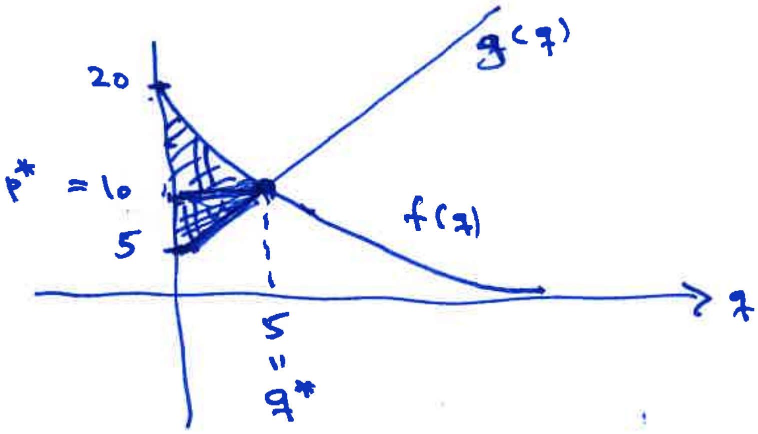


$p = g(q)$ omvendt tilbudsfunksjon

$p = f(q)$ omvendt etterspørselsfn.

$$\boxed{\begin{array}{ll} \text{Konsumentoverskudd:} & CS = \int_0^{q^*} f(q) - p^* \, dq \\ \text{Produsent overskudd:} & PS = \int_0^{q^*} p^* - g(q) \, dq \end{array}}$$

Eks: $f(q) = \dfrac{100}{q+5}$ $\qquad g(q) = q+5$



$$\frac{100}{q+5} = q+5$$

$$100 = (q+5)^2$$

$$q+5 = \pm\sqrt{100} = \pm 10$$

$$q = 10-5 = 5 \quad \text{eller} \quad \cancel{q = -10-5 = -15}$$

$$CS = \int_0^5 \frac{100}{q+5} - 10 \, dq = \Big[100 \cdot \ln(q+5) - 10q\Big]_0^5$$

$$= \big(100 \ln(10) - 50\big) - \big(100 \cdot \ln 5 - 0\big) = 100\big(\ln(10) - \ln 5\big) - 50$$

$$= 100 \cdot \ln \frac{10}{5} - 50 = 100 \ln 2 - 50 \simeq 19$$

$$PS = \int_0^5 10 - (q+5) \, dq = \int_0^5 5 - q = \Big[5q - \tfrac{1}{2}q^2\Big]_0^5$$

$$= \big(25 - \tfrac{1}{2} \cdot 25\big) - 0 = \frac{25}{2} = 12.5$$

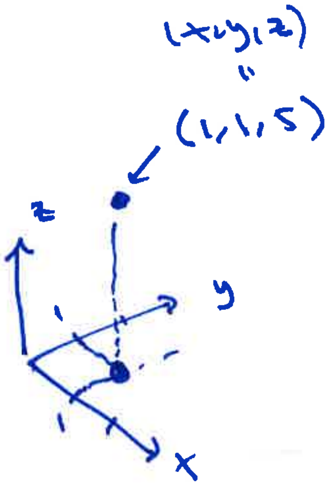
2 Partiell derivasjon

$f(x,y)$: funksjon i to variabler

Eks: $f(x,y) = 5 + x - y$

$f(x,y) = x^3 - 3xy + y^2$

$f(x,y) = e^{xy}$



Graf: 3-dimensjonal

$z = f(x,y)$

$(x,y) \longrightarrow f \longrightarrow z$

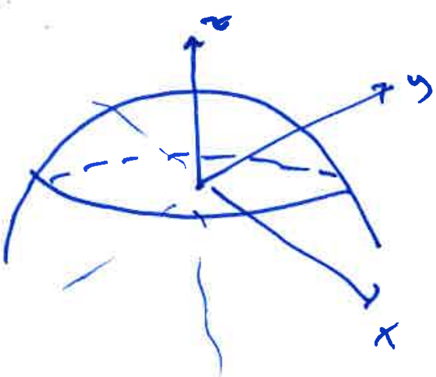
input output

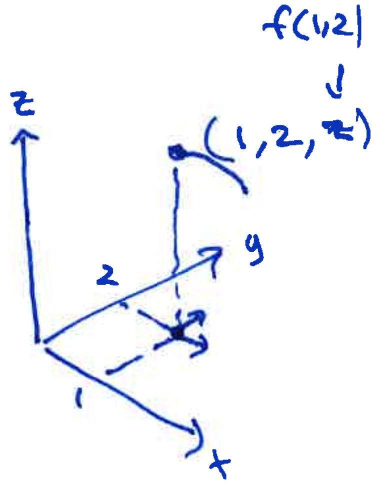
Eks:

$f(x,y) = 5 + x - y$

$f(1,1) = 5$

Graf = Alle punkt (x,y,z) slik at (x,y) i D_f og $z = f(x,y)$





Partiell derivasjon:

$$f'_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f'_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

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Eks: $f(x,y) = x^3 - 3xy + y^3$

$f'_x = 3x^2 - 3y$ ← y er konstant, deriver mhp. x

$f'_y = -3x + 3y^2$ ← x er konstant, deriver mhp. y

$f(x,y) = e^{xy} = e^u$, $u = xy$

$f'_x = e^u \cdot u'_x = y \cdot e^{xy}$

$f'_y = e^u \cdot u'_y = x \cdot e^{xy}$

Del 2:

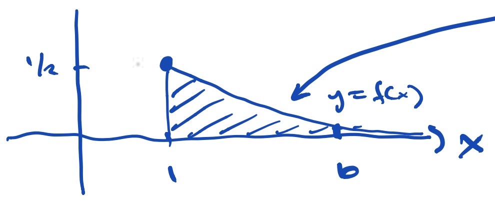
Repetisjon:

$\int_{t_1}^{t_2} f(x)\,dx$ ← samlet kontantstrøm

$\int_{t_1}^{t_2} f(x)\,e^{-rx}\,dx$ ← samlet nåverdi av kontantstrøm

Oppgaver: Oppgaveark 20

5 d) $\int_1^\infty \frac{1}{x^2+x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2+x}\,dx$



$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad \Big| \cdot (x)(x+1)$

$1 = A(x+1) + Bx$

$1 = \underbrace{(A+B)}_{0}x + \underbrace{A}_{1}$

$A = 1$
$B = -1$

$\int_1^b \frac{1}{x^2+x}\,dx = \int_1^b \frac{1}{x} - \frac{1}{x+1}\,dx$

$= \left[\ln|x| - \ln|x+1|\right]_1^b$

$= \left[\ln \frac{|x|}{|x+1|}\right]_1^b = \ln\left(\frac{b}{b+1}\right) - \ln\left(\frac{1}{2}\right) \xrightarrow{b\to\infty} \underbrace{\ln 1}_{0} - \underbrace{\ln\left(\frac{1}{2}\right)}_{-\ln(2^{-1})}$

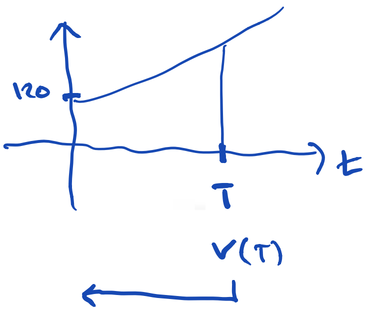
$= \ln 2 \approx 0.69$

Alt: $= (\ln b - \ln(b+1)) - (\ln 1 - \ln 2)$
$= \ln b - \ln(b+1) + \ln 2$

Ubestemte uttrykk: "$\frac{0}{0}$" "$\frac{\infty}{\infty}$" "$\infty - \infty$" "$\infty \cdot 0$"

7. $V(t) = 120\, e^{\sqrt{t}/5}$ verdien etter t år

a) Nåverdi:



Salgssum: $V(T) = 120 e^{\sqrt{T}/5}$ $r = 0.04$

nåverdi: $V(T) \cdot e^{-0.04T}$

$$N(T) = 120\, e^{\sqrt{T}/5} e^{-0.04T} = 120\, e^{\overbrace{\sqrt{T}/5 - 0.04T}^{u}}$$

max $N(T)$:

$$N'(t) = 120 e^u \cdot u'$$

$$= \underbrace{120 e^u}_{\text{pos.}} \cdot \left(\frac{1}{5} \cdot \frac{1}{2\sqrt{T}} - 0.04 \right)$$

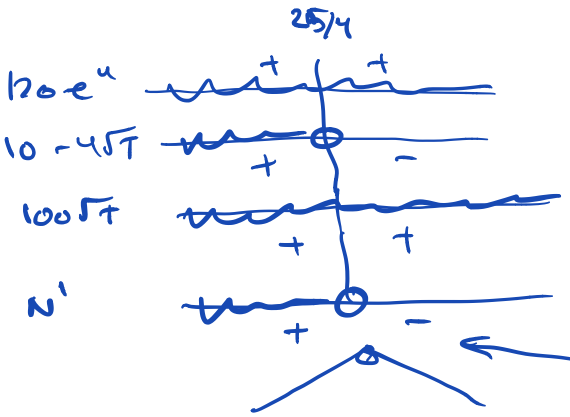
$$= 120 e^u \left(\frac{1 \cdot 10}{10\sqrt{T} \cdot 10} - \frac{4}{100} \frac{\sqrt{T}}{\sqrt{T}} \right)$$

$$= 120 e^u \left(\frac{10 - 4\sqrt{T}}{100\sqrt{T}} \right) = 0$$

$$10 - 4\sqrt{T} = 0$$

$$4\sqrt{T} = 10$$

$$\sqrt{T} = \frac{10}{4} = \frac{5}{2} \qquad T = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$$



maks nåverdi $= 6.25$

b) Første dobling:

$V(T) = 240$

$$\frac{120\, e^{\sqrt{t}/5}}{120} = \frac{240}{120}$$

$$e^{\sqrt{t}/5} = 2 \quad | \ln(\cdot)$$

$$\sqrt{t}/5 = \ln 2$$

$$\sqrt{t} = 5 \ln 2$$

$$t = (5 \ln 2)^2$$

$$T = \underline{\underline{25 \cdot (\ln 2)^2}}$$

$$\approx 12 \text{ år}$$

$\ln 4 = \ln 2^2$
$= 2 \ln 2$

Andre dobling:

$V(t) = 480$

$$\frac{120\, e^{\sqrt{t}/5}}{120} = \frac{480}{120}$$

$$e^{\sqrt{t}/5} = 4$$

$$\sqrt{t}/5 = \ln 4$$

$$\sqrt{t} = 5 \ln 4$$

$$t = (5 \ln 4)^2$$

$$= 25 \cdot (\ln 4)^2$$

$$= 25 \cdot (2 \ln 2)^2$$

$$= 100 \cdot (\ln 2)^2$$

$$= 4T$$

Fra første til andre dobling: $4T - T = \underline{\underline{3T}} \approx 36$ år

5a $\int \frac{3x-4}{x^2+x}\,dx = \int \frac{-4}{x} + \frac{7}{x+1}\,dx$

$= -4\ln|x| + 7\ln|x+1| + C$

$= \ln|x+1|^7 - \ln|x|^4 + C$

$= \ln \frac{|x+1|^7}{|x|^4} + C$

$\frac{3x-4}{x\cdot(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad | \cdot x(x+1)$

$3x-4 = A(x+1) + Bx$

$3x-4 = (A+B)x + A$

$A+B = 3$, $A = -4$

$A = -4$

$A+B=3$

$-4+B=3$

$B=7$

b) $\int \underbrace{18x^2}_{u'}\cdot\underbrace{\ln(x+1)}_{v}\,dx = 6x^3\ln(x+1) - \int 6x^3\cdot\frac{1}{x+1}\,dx$

$\boxed{u = 6x^3 \quad v = \ln(x+1) \quad u' = 18x^2 \quad v' = \frac{1}{x+1}}$

$\boxed{u = x+1 \quad du = dx}$

$= 6x^3\ln(x+1) - 6\int \frac{x^3}{x+1}\,dx$

$x^3 : (x+1) = x^2 - x + 1$

$-(x^3+x^2)$

$-x^2$

$-(-x^2-x)$

x

$-(x+1)$

-1

$= 6x^3\ln(x+1) - 6\int x^2 - x + 1 + \frac{-1}{x+1}\,dx$

$= 6x^3\ln(x+1) - 6\left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \ln|x+1|\right) + C$

$= 6x^3\ln(x+1) - 2x^3 + 3x^2 - 6x + 6\ln|x+1| + C$

c. $\int e^{\sqrt{x}}\,dx = \int e^u\cdot(2\sqrt{x})\,du = \int 2ue^u\,du$

$\boxed{u = \sqrt{x} \quad du = \frac{1}{2\sqrt{x}}\,dx}$

delvis integrasjon

$= 2(ue^u - e^u) + C$

$= 2ue^u - 2e^u + C$

$= 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$