Plan

1 Substitusjon
2 Delvis integrasjon
3 Integrasjon av brøkuttrykk

① Substitusjon:

$$\int f(x)\,dx = \int g(u)\,du$$

$$\boxed{\begin{aligned} u &= u(x) \\ du &= u'(x)\cdot dx \end{aligned}}$$

$$dx = \frac{du}{u'}$$

Oppgavearh. 17.

9c $\int x e^{-x^2} dx = \int \cancel{x}\cdot e^u \frac{du}{-2\cancel{x}}$

$$\boxed{\begin{aligned} u &= -x^2 \\ du &= -2x\,dx \end{aligned}}$$

$$dx = \frac{du}{-2x}$$

$$= \int -\frac{1}{2}e^u\,du = -\frac{1}{2}\cdot e^u + C = -\frac{1}{2}e^{-x^2} + C$$

e: $\int \frac{\ln x}{x}\,dx = \int \frac{u}{\cancel{x}}\cancel{x}\,du = \int u\,du$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$$

$$dx = x\cdot du$$

$$= \frac{1}{2}u^2 + C$$

$$= \frac{1}{2}(\ln x)^2 + C$$

<u>Lo.</u>

$$\int \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^u}{\cancel{\sqrt{x}}}(-2\cancel{\sqrt{x}})\,du$$

$$\boxed{\begin{aligned} u &= 1-\sqrt{x} \\ du &= -\frac{1}{2\sqrt{x}}\,dx \end{aligned}}$$

$$dx = -2\sqrt{x}\,du$$

$$= \int -2e^u\,du = -2e^u + C = \underline{\underline{-2e^{1-\sqrt{x}} + C}}$$

<u>Eks:</u>

$$\int e^{\sqrt{x}}\,dx = \int e^u\, 2\sqrt{x}\,du$$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \end{aligned}}$$

$$dx = 2\sqrt{x}\,du$$

$$= \int 2\sqrt{x}\,e^u\,du = \int \underline{2ue^u\,du}$$

$$= 2\int ue^u\,du = 2(ue^u - e^u + C)$$

delvis integrasjon

$$= 2ue^u - 2e^u + C$$

$$= \underline{\underline{2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}}$$

(2) Delvis integrasjon:

Eks: $\int \underbrace{x}_{u'} \cdot \underbrace{\ln x}_{v} \, dx$

$= uv - \int uv' dx$

$u = \frac{1}{2}x^2 \qquad v = \ln x$

$u' = x \qquad v' = 1/x$

$= \frac{1}{2}x^2 \cdot \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x} \, dx$

$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2} x \, dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C$

$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

Derivasjon:

$(u \cdot v)' = u' \cdot v + u \cdot v'$

Delvis integrasjon = produktregelen for derivasjon baklengs:

$(uv)' = u' \cdot v + u \cdot v'$

$uv = \int u'v \, dx + \int uv' dx$

$$\boxed{\int u'v \, dx = uv - \int uv' dx}$$

Eks: $\int \underbrace{x}_{u'} \cdot \underbrace{e^x}_{✓} dx = \cancel{\frac{1}{2}x^2 \cdot e^x - \int \frac{1}{2}x^2 e^x dx}$

$\cancel{u = x^2/2 \qquad v = e^x}$

$\cancel{u' = x \qquad v' = e^x}$

$\int \underbrace{x}_{v} \cdot \underbrace{e^x}_{u'} dx = e^x \cdot x - \int e^x \cdot 1 \, dx$

$u = e^x \qquad v = x$

$u' = e^x \qquad v' = 1$

$= xe^x - \int e^x dx = xe^x - e^x + C$

Formel: $\boxed{\int u'v\,dx = uv - \int uv'\,dx}$

$$\int \ln x\,dx = \int \underbrace{1}_{u'} \cdot \underbrace{\ln x}_{v}\,dx = uv - \int uv'\,dx$$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

$$= x \cdot \ln x - \int x \cdot \frac{1}{\cancel{x}}\,dx$$

$$= x \cdot \ln x - \int 1\,dx = \underline{\underline{x \ln x - x + C}}$$

$$(x \ln x - x)' = \underline{1 \cdot \ln x} + \cancel{x \cdot \frac{1}{x}} - \cancel{1} = \underline{\underline{\ln x}}$$

Viktig: Se på graden til nevneren!

③ Integrasjon av brøkuttrykk

Eks: $\int \frac{1}{1-x}\,dx \qquad \int \frac{x}{1-x}\,dx \qquad \int \frac{1}{1-x^2}\,dx$

Eks:

$$\int \frac{1}{1-x}\,dx = \int \frac{1}{u} \cdot \frac{du}{-1} = \int \frac{1}{-u}\,du = -\int \frac{1}{u}\,du$$

$$\boxed{\begin{array}{l} u = 1-x \\ du = (-1) \cdot dx \end{array}}$$

$$dx = \frac{du}{-1}$$

$$= -\ln|u| + C$$

$$= \underline{\underline{-\ln|1-x| + C}}$$

$$\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C$$ når A, a, b konstanter, $a \neq 0$

Ek:

$$\int \frac{x}{1-x}\,dx \xrightarrow[\text{polynom-divisjon}]{} = \int -1 + \frac{1}{1-x}\,dx = -x + \int \frac{1}{1-x}\,dx$$

$$\begin{array}{l} x : (-x+1) = -1 \\ -(x-1) \\ \hline \quad 1 \end{array} \quad \Bigg\} \quad \frac{x}{1-x} = -1 + \frac{1}{1-x}$$

$$= -x + (-\ln|1-x| + C)$$

$$= \underline{\underline{-x - \ln|1-x| + C}}$$

$$\int \frac{1}{1-x}\,dx = \quad \boxed{\begin{array}{l} u = 1-x \\ du = -1 \cdot dx \end{array}} \quad \int \frac{1}{u}\,\frac{du}{-1}$$

$$= \int \frac{1}{-u}\,du = -\int \frac{1}{u}\,du = -\ln|u| + C$$

$$= \underline{-\ln|1-x| + C}$$

Delbrøksoppspalting:

Eks:

$$\int \frac{2}{1-x^2}\,dx = \cancel{2\ln|1-x^2| + C}$$

$$\int \frac{1}{1-x} + \frac{1}{1+x}\,dx = \underline{-\ln|1-x| + \ln|1+x| + C}$$

Delbrøksoppspalting:

$$\frac{2}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+X}$$

: 1 Faktoriser nevner:

$$1-x^2 = (1-x)(1+x)$$

2

$$\frac{2}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+X}$$

Må finne konstanter A, B slik at dette stemmer.

3 Multipliser med fellesnevner

$$(1-x)(1+x) = 1-x^2$$

$$2 = \frac{A}{\cancel{1-x}}(\cancel{1-x})(1+x) + \frac{B}{\cancel{1+x}}\cdot(1-x)(\cancel{1+x})$$

$$2 = A\cdot(1+x) + B(1-x)$$

$$= A + Ax + B - Bx$$

$$= (Ax - Bx) + (A+B)$$

$$\underbrace{2}_{f(x)} = \underbrace{(A-B)x + (A+B)}_{g(x)}$$

$$\boxed{\begin{aligned} A - B &= 0 \\ A + B &= 2 \end{aligned}}$$

$$B = A \qquad A + A = 2 \qquad 2A = 2 \qquad A = 1 \qquad B = 1$$

$$\boxed{\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+X}}$$

Alt: $2 = A(1+x) + B(1-x)$

$x=-1$: $2 = A\cdot 0 + B\cdot 2 \Rightarrow B = 1$

$x=1$: $2 = A\cdot 2 + B\cdot 0$ $\quad A = 1$

Del 2:

Repetisjon:

* Delvis integrasjon: $\int u'v\,dx = uv - \int uv'\,dx$
* Delbrøksoppspalting:

$$\boxed{\frac{cx+d}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}} \qquad a \neq b$$

Andre tilfeller:

$$\frac{cx+d}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$$

$\frac{2}{5}x^{5/2}$ $=$ $\frac{x^{5/2}}{5/2}$

$u = \frac{x^{5/2}}{5/2}$ $\quad v = \ln x$

$u' = x^{3/2}$ $\quad v' = 1/x$

Oppgaver, Oppgaveark 18

2d. $\int x\sqrt{x}\,\ln x\,dx = \int x^{3/2}\cdot \ln x\,dx$

$= \frac{2}{5}x^{5/2}\cdot\ln x - \int \frac{2}{5}x^{5/2}\cdot\frac{1}{x}\,dx = \frac{2}{5}x^{5/2}\ln x - \frac{2}{5}\int x^{3/2}\,dx$

$= \frac{2}{5}x^{5/2}\ln x - \frac{2}{5}\left(\frac{2}{5}x^{5/2}\right) + C = \frac{2}{5}x^2\sqrt{x}\,\ln x - \frac{4}{25}\cdot x^2\sqrt{x} + C$

3d. $\int x^2 e^x\,dx = e^x\cdot x^2 - \int e^x\cdot 2x\,dx$

$\boxed{\begin{matrix} u = e^x & v = x^2 \\ u' = e^x & v' = 2x \end{matrix}}$

$= x^2e^x - \int 2xe^x\,dx$

$\boxed{\begin{matrix} u = e^x & v = 2x \\ u' = e^x & v' = 2 \end{matrix}}$

$= x^2e^x - \left(2xe^x - \int 2e^x\,dx\right)$

$\rightarrow = x^2e^x - 2xe^x + 2e^x + C$

4d. $\int \frac{x^2}{4-x^2}\,dx = \int -1 + \frac{4}{4-x^2}\,dx = -x + \int \frac{4}{4-x^2}\,dx$

$x^2 : (-x^2+4) = -1$

$-(x^2-4)$

$\quad 4$

$\frac{4}{4-x^2} = \frac{A}{2-x} + \frac{B}{2+x} \quad \Big|\cdot(2-x)(2+x)$

$4-x^2 = (2-x)(2+x) = -(x-2)(x+2)$

$4 = A(2+x) + B(2-x)$

$4 = (A-B)x + (2A+2B)$

$A - B = 0 \qquad 2A+2B = 4$

$A = B \qquad 4A = 4$

$A = 1 = B$

$$= -x + \int \frac{1}{2-x} + \frac{1}{2+x}\,dx = -x - \ln|2-x| + \ln|2+x| + C$$

$$= -x + \ln\left|\frac{2+x}{2-x}\right| + C$$

$\ln a - \ln b = \ln \frac{a}{b}$

$\ln a + \ln b = \ln(ab)$

5g. $$\int \frac{x^2}{(1-x)^2}\,dx = \int \frac{x^2}{1-2x+x^2} = \int 1 + \frac{2x-1}{x^2-2x+1}\,dx$$

$x^2 : (x^2-2x+1) = 1$

$-(x^2-2x+1)$

$2x-1$

$$= x + \int \frac{2x-1}{(x-1)^2}\,dx$$

$\frac{2x-1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \quad | \cdot (x-1)^2$

$\boxed{2x-1 = A\cdot(x-1) + B}$

$2x-1 = \underset{\underset{2}{\shortparallel}}{A}x + \underset{\underset{-1}{\shortparallel}}{(-A+B)}$

$A = 2$

$-A+B = -1$

$-2+B = -1$

$B = 1$

$$= x + \int \frac{2}{x-1} + \frac{1}{(x-1)^2}\,dx = x + 2\ln|x-1| + \int (x-1)^{-2}\,dx$$

$n = -2$

$n+1 = -1$

$$= x + 2\ln|x-1| + \frac{(x-1)^{-1}}{-1} + C$$

$$= x + 2\ln|x-1| - \frac{1}{x-1} + C$$

$\tan x = \frac{\sin x}{\cos x}$

arctan(x): $$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

arctan x = invers tan x

q. $$\int \frac{\sqrt{x}+1}{1-\sqrt{x}}\,dx = \int \frac{\sqrt{x}+1}{u}\cdot(-2\sqrt{x})\,du$$

$\boxed{\sqrt{x} = 1-u}$

$u = 1-\sqrt{x}$

$du = -\frac{1}{2\sqrt{x}}\,dx$

$dx = -2\sqrt{x}\,du$

$$= \int \frac{-2\sqrt{x}(\sqrt{x}+1)}{u}\,du$$

$$= \int \frac{-2(1-u)(2-u)}{u}\,du$$

$$= \int \frac{-2(u^2-3u+2)}{u}\,du = \int -2u + 6 - \frac{4}{u}\,du$$

$$= -u^2 + 6u - 4\ln|u| + C = -(1-\sqrt{x})^2 + 6(1-\sqrt{x}) - 4\ln|1-\sqrt{x}| + C$$

$$= 5 - 4\sqrt{x} - x - 4\ln|1-\sqrt{x}| + C$$

$$= -4\sqrt{x} - x - 4\ln|1-\sqrt{x}| + C$$

6b. $\int x \ln(1-x)\,dx = \frac{1}{2}x^2 \ln(1-x) - \int \frac{x^2}{2}\cdot\frac{-1}{1-x}\,dx$

$\boxed{\begin{array}{ll} u = x^2/2 & v = \ln(1-x) \\ u' = x & v' = \frac{1}{1-x}\cdot(-1) \end{array}}$

$= \frac{1}{2}x^2 \ln(1-x) + \frac{1}{2}\int \frac{x^2}{1-x}\,dx = \frac{1}{2}x^2 \ln(1-x) + \frac{1}{2}\int -x-1+\frac{1}{1-x}\,dx$

$= \frac{1}{2}x^2 \ln(1-x) + \frac{1}{2}\left(-\frac{1}{2}x^2 - x - \ln|1-x|\right) + C$

$= \frac{1}{2}x^2 \ln(1-x) - \frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{2}\ln|1-x| + C$

$x^2 : (-x+1) = -x-1$
$-(x^2-x)$
x
$-(x-1)$
1

7. $\int 2x^3 e^{-x^2}\,dx = \int 2x^3 e^u \cdot \frac{du}{-2x} = \int \frac{\cancel{2}x^{\cancel{3}\,2} e^u}{-\cancel{2x}}\,du$

$\boxed{\begin{array}{l} u = -x^2 \\ du = -2x\,dx \end{array}}$

$= \int -x^2 e^u\,du = \int u e^u\,du$

$\boxed{\begin{array}{ll} v = e^u & w = u \\ v' = e^u & w' = 1 \end{array}}$

$= e^u \cdot u - \int \underbrace{e^u \cdot 1}_{e^u}\,du = ue^u - e^u + C$

$= -x^2 e^{-x^2} - e^{-x^2} + C$

8. $\int \sqrt{x}\, e^{\sqrt{x}}\,dx = \int \sqrt{x}\, e^u \cdot (2\sqrt{x})\,du = \int 2(\sqrt{x})^2 e^u\,du$

$\boxed{\begin{array}{l} u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\cdot dx \end{array}}$

$= \int 2u^2 e^u\,du = \int 2x e^u\,du$

$= 2\cdot \int u^2 e^u\,du$

3d

$= 2\left(u^2 e^u - 2ue^u + 2e^u\right) + C$

$= 2xe^{\sqrt{x}} - 4\sqrt{x}e^{\sqrt{x}} + 4e^{\sqrt{x}} + C$

10. Eksamen 05/2018 Oppg. 1.

a) $f(x) = 0.6\ln(1+x) + 0.4\ln(1-x)$, $0 \le x < 1$

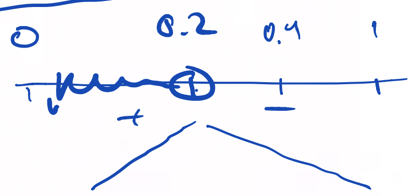
maks: $x = x^*$; $f(x^*)$

$$f'(x) = 0.6\cdot\frac{1}{1+x}\cdot 1 + 0.4\cdot\frac{1}{1-x}\cdot(-1) = \frac{0.6}{1+x} - \frac{0.4}{1-x}$$

$$= \frac{0.6(1-x) - 0.4(1+x)}{(1+x)(1-x)} = \frac{0.2 - x}{(1+x)(1-x)}$$

$f'(x) = 0$: $x^* = 0.2$

maxpkt.



$$f(x^*) = f(0.2)$$
$$= 0.6\ln(1.2) + 0.4\ln(0.8) \simeq 0$$

b) f konveks $\Leftrightarrow$ $f''(x) \ge 0$

f konkav $f''(x) \le 0$

$$f''(x) = \Big(\underbrace{0.6(1+x)^{-1} - 0.4(1-x)^{-1}}_{f'(x)}\Big)' = 0.6\cdot(-1)(1+x)^{-2}\cdot 1$$
$$- 0.4\cdot(-1)(1-x)^{-2}\cdot(-1) = \frac{-0.6}{(1+x)^2} + \frac{-0.4}{(1-x)^2} \le 0$$

for alle x

$\Downarrow$

f konkav $\frown$

c) $f(x) < 0$ når $x \ge 2x^* = 2\cdot 0.2 = 0.4$

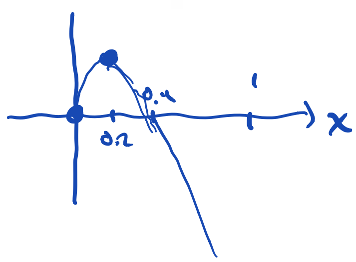
f avtagende når $x \ge 0.4$

$f(0.4) = 0.6\ln 1.4 + 0.4\ln 0.6 \simeq -0.02 < 0$

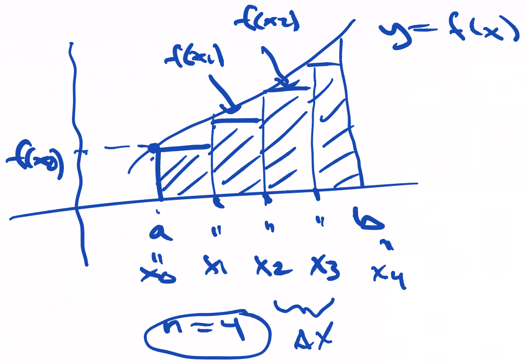
$\Downarrow$

$f(x) < 0$ for $x \ge 0.4$

d)



Riemann summer:



f(x) kont. på [a,b]

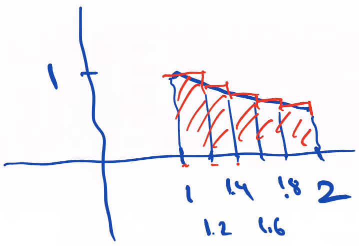
⇒ Deler [a,b] i n like delintervall

$$x_i = a + (i-1)\cdot \underbrace{\frac{b-a}{n}}_{\Delta x}$$

Riemannsum med støttepunkt i hvert delintervall:

$$\Delta x \cdot f(x_0) + \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \Delta x \cdot f(x_3)$$

Eks: $f(x) = 1/x$ på $[1,2]$ $\quad n = 5 \Rightarrow \Delta x = \frac{b-a}{n} = \frac{2-1}{5} = 0.2$



Riemannsum:

f(1) f(1,2)

$$\Delta x \cdot f(x_0) + \Delta x \cdot f(x_1) + \dots$$

$$= 0.2 \cdot 1 + 0.2 \cdot \frac{1}{1.2} + 0.2 \cdot \frac{1}{1.4} + 0.2 \cdot \frac{1}{1.6} + 0.2 \cdot \frac{1}{1.8}$$

$$= 0.2 + \frac{0.2}{1.2} + \frac{0.2}{1.4} + \frac{0.2}{1.6} + \frac{0.2}{1.8}$$