Plan

1 Regning med matriser
2 Eksamensoppgave 05/2017

Repetisjon:

1) Matrisemultiplikasjon

$$\underset{m\times n}{A} \cdot \underset{n\times p}{B} \rightsquigarrow \underset{m\times p}{AB}$$

(definert når # kolonner i A = # rader i B)

Merk: $AB \neq BA$

Eks:

$$\underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}} \cdot \underset{3\times 3}{\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}} = \underset{2\times 3}{\underline{\underline{\begin{pmatrix} -1 & 6 & 11 \\ -1 & 1 & -2 \end{pmatrix}}}}$$

2) Transponering:

$$\underset{m\times n}{A} \rightsquigarrow \underset{n\times m}{A^T}$$

Eks:

$$\underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}}^T = \underset{3\times 2}{\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & -1 \end{pmatrix}}$$

A kalles symmetrisk hvis $A^T = A$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 0 \end{pmatrix}$$

Symmetrisk

3) Inverse matriser

Defn: Matrisen $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ kalles identitetsmatrisen. Den har egenskapen at

$$\boxed{A \cdot I = A \quad \text{og} \quad I \cdot A = A}$$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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Defn: En invers matrise A^{-1} til A er en matrise slik at

$$\boxed{A^{-1} \cdot A = I, \quad \text{og} \quad A \cdot A^{-1} = I}$$

Resultat:

1) A^{-1} fins $\Longleftrightarrow |A| \neq 0$
2) Når $|A| \neq 0$, så er A^{-1} entydig
3) $A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A)$ når $|A| \neq 0$

$$= \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T$$

Anvendelse: Alternativ til Kramers regel

Anta at et kvadratisk lineært system $A\underline{x} = \underline{b}$ slik at $|A| \neq 0$. Da har vi:

$$A \cdot \underline{x} = \underline{b}$$

A^{-1} fins siden $|A| \neq 0$. Mult. likn. med A^{-1} fra venstre

$$A^{-1} \cdot A\underline{x} = A^{-1} \cdot \underline{b}$$

$$I\underline{x} = A^{-1} \cdot \underline{b}$$

$$\boxed{\underline{x} = A^{-1} \cdot \underline{b}} \longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Oppgaveark 24:

2 f) $A = \begin{pmatrix} 7 & 1 & 4 \\ -2 & 1 & -2 \\ 3 & 3 & 0 \end{pmatrix}$ A^{-1} : fins ikke

$|A| = 3\cdot(-2-4) - 3(-14+8)$
$= -18+18 = \underline{0}$

e) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ $A^{-1} = \frac{1}{4}\cdot \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}^T$

$|A| = 2\cdot 3 - 1\cdot 1 + 1\cdot(-1)$
$= 6-2 = \underline{4} \neq 0$ $= \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$

$C_{11} = 3$ $C_{12} = -1$ $C_{13} = -1$
$C_{21} = -1$ $C_{22} = 3$ $C_{23} = -1$
$C_{31} = -1$ $C_{32} = -1$ $C_{33} = 3$

3. $A = \begin{pmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{pmatrix}$ $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\underline{b} = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}$ $\boxed{A\cdot \underline{x} = b}$

b) $|A| = \begin{vmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{vmatrix} = +t(t^2-1) = \underline{t(t+1)(t-1)}$

$|A| = 0$: $t = 0, \pm 1$ → ingen eller uendelig mange
$|A| \neq 0$: $t \neq 0, \pm 1$ → én løsning

$\underline{t=0}$: $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ uendelig mange løsn.

$t=1$: $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ uendelig mange løsn.

$t=-1$: $\left(\begin{array}{ccc|c} -1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right)$ ingen løsn.

$$\begin{cases} t=-1: & \text{ingen løsn.} \\ t=0,1: & \text{uendelig mange løsn.} \\ t\neq 0,\pm 1: & \text{én løsn} \end{cases}$$

c) A^{-1} eksisterer når $|A| = t(t+1)(t-1) \neq 0$, dvs $t \neq 0, \pm 1$

$t \neq 0, \pm 1$:

$$A = \begin{pmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{pmatrix} \qquad A^{-1} = \frac{1}{t(t^2-1)} \cdot \begin{pmatrix} t^2 & 0 & -t \\ 0 & t^2-1 & 0 \\ -t & 0 & t^2 \end{pmatrix}^T$$

$$= \frac{1}{t(t^2-1)} \cdot \begin{pmatrix} t^2 & 0 & -t \\ 0 & t^2-1 & 0 \\ -t & 0 & t^2 \end{pmatrix}$$

$$A \cdot \underline{x} = \underline{b}$$

$$A^{-1} A \underline{x} = A^{-1} \underline{b}$$

$$\underline{x} = A^{-1} \cdot \underline{b} = \underbrace{\frac{1}{t(t^2-1)} \begin{pmatrix} t^2 & 0 & -t \\ 0 & t^2-1 & 0 \\ -t & 0 & t^2 \end{pmatrix}}_{A^{-1}} \cdot \underbrace{\begin{pmatrix} t \\ 0 \\ t \end{pmatrix}}_{\underline{b}}$$

$$= \frac{1}{t(t^2-1)} \cdot \begin{pmatrix} t^3 - t^2 \\ 0 \\ t^3 - t^2 \end{pmatrix} = \frac{1}{t(t+1)(t-1)} \cdot \begin{pmatrix} t^2(t-1) \\ 0 \\ t^2(t-1) \end{pmatrix}$$

$$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{t^2\cancel{(t-1)}}{\cancel{t}(t+1)\cancel{(t-1)}} \\ 0 \\ \frac{t^2\cancel{(t-1)}}{\cancel{t}(t+1)\cancel{(t-1)}} \end{pmatrix} = \underline{\underline{\begin{pmatrix} \frac{t}{t+1} \\ 0 \\ \frac{t}{t+1} \end{pmatrix}}}$$

5.

c) $A \cdot (3B - C) + (A - 2B)C + 2B(C + 2A)$

$= A \cdot 3 \cdot B - \cancel{AC} + \cancel{AC} - \cancel{2BC} + \cancel{2BC} + 2B \cdot 2A$

$= \underline{\underline{3AB + 4BA}}$

e) $(BAB^{-1})^2 \cdot B^2 = (BAB^{-1}) \cdot (BAB^{-1}) \cdot B^2$

$= BA\cancel{B^{-1}}\cancel{B}A\cancel{B^{-1}}\cancel{B} \cdot B = BAAB = \underline{\underline{BA^2B}}$

f) $(A-B)(C-A) + (C-B)(A-C) + (C-A)^2$ ← $(C-A)(C-A)$

$= \cancel{AC} - \cancel{BC} - \cancel{A^2} + \cancel{BA} + \cancel{CA} - \cancel{BA} - \cancel{C^2} + \cancel{BC}$

$+ \cancel{C^2} - \cancel{AC} - \cancel{CA} + \cancel{A^2} = \underline{\underline{0}}$

1 Regning med matriser

Noen viktige regneregler:

$AB \neq BA$!

a) Determinant:

i) $|A \cdot B| = |A| \cdot |B|$

ii) $|c \cdot A| = c^n \cdot |A|$ når A er $n \times n$-matrise

iii) $|A^T| = |A|$

iv) $|A^{-1}| = \frac{1}{|A|}$

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$|A| = 1 \cdot 4 - 2 \cdot 3$

$5 \cdot A = \begin{pmatrix} 5 \cdot 1 & 5 \cdot 2 \\ 5 \cdot 3 & 5 \cdot 4 \end{pmatrix}$

$|5 \cdot A| = (5 \cdot 1) \cdot (5 \cdot 4) - (5 \cdot 3) \cdot (5 \cdot 2)$

$= 5^2 \cdot (1 \cdot 4 - 3 \cdot 2)$

$A^{-1} \cdot A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$|A^{-1} \cdot A| = |I|$

$|A^{-1}| \cdot |A| = 1$

$\Downarrow$

$|A^{-1}| = \frac{1}{|A|}$

b) Transponering:

i) $(A^T)^T = A$

ii) $(A \pm B)^T = A^T \pm B^T$

iii) $(A \cdot B)^T = B^T \cdot A^T$

c) Inverse matriser:

i) $(A^{-1})^{-1} = A$

ii) $(AB)^{-1} = B^{-1} \cdot A^{-1}$

$(AB)(B^{-1}A^{-1})$

$= A \cancel{B} \cancel{B^{-1}} A^{-1}$

$= AA^{-1} = I$

Oppg. 6. $|A| = 2 \qquad |B| = -5 \qquad A, B: 3 \times 3$

a) $|AB| = |A| \cdot |B| = 2 \cdot (-5) = \underline{\underline{-10}}$

b) $|3A| = 3^3 \cdot |A| = 27 \cdot 2 = \underline{\underline{54}}$

c) $|-2B^T| = (-2)^3 \cdot |B^T| = -8 \cdot |B| = -8 \cdot (-5) = \underline{\underline{40}}$

d) $|2A^{-1}B| = 2^3 \cdot |A^{-1}| \cdot |B| = 8 \cdot \frac{1}{2} \cdot (-5) = \underline{\underline{-20}}$

Eksamen 05/2017, Oppg. 1.

a) $a=1$

$$\left(\begin{array}{ccc|c} 2 & 2 & 0 & 4 \\ 2 & 2 & 2 & 1 \\ 0 & 2 & 2 & 2 \end{array}\right) \begin{matrix} \\ -1 \\ \\ \end{matrix} \to \left(\begin{array}{ccc|c} 2 & 2 & 0 & 4 \\ 0 & 0 & 2 & -3 \\ 0 & 2 & 2 & 2 \end{array}\right) \updownarrow$$

$$\to \left(\begin{array}{ccc|c} 2 & 2 & 0 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & -3 \end{array}\right) \qquad \begin{aligned} 2x + 2y &= 4 \\ 2y + 2z &= 2 \\ 2z &= -3 \end{aligned}$$

$z = \underline{\underline{-3/2}}$

$2y = 2 - 2z = 2 - 2(-3/2) = 5$

$y = \underline{\underline{5/2}}$

$2x = 4 - 2y = 4 - 2 \cdot (5/2) = -1$

$x = \underline{\underline{-1/2}}$

En løsn: $(x, y, z) = \underline{\underline{(-1/2, 5/2, -3/2)}}$

b) $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$ $a=1$

$|A| = 2\cdot(4-4) - 2(4-0) = \underline{-8} \neq 0$

$A^{-1} = \frac{1}{|A|}\cdot \mathrm{adj}(A) = \frac{1}{-8}\cdot \begin{pmatrix} 0 & -4 & 4 \\ -4 & 4 & -4 \\ 4 & -4 & 0 \end{pmatrix}^T = \frac{1}{-8}\begin{pmatrix} 0 & -4 & 4 \\ -4 & 4 & -4 \\ 4 & -4 & 0 \end{pmatrix}$

$A^T = A \Rightarrow C^T = C$
(A symm.) (C symm.)

$= \underline{\underline{\begin{pmatrix} 0 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 0 \end{pmatrix}}} = \underline{\underline{\frac{1}{2}\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}}}$

$\underline{b}$ med $a=1$

$A\cdot \underline{x} = \underline{b}$

$A^{-1}A\underline{x} = A^{-1}\underline{b}$

$\underline{x} = A^{-1}\underline{b} = \underbrace{\frac{1}{2}\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}}_{A^{-1}}\cdot \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$

$= \frac{1}{2}\cdot\begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -1/2 \\ 5/2 \\ -3/2 \end{pmatrix}}}$

c) $A\underline{x} = \underline{b}$ har én løsn. $\Leftrightarrow$ $|A| \neq 0$

$|A| = \begin{vmatrix} 1+a & 2 & 1-a \\ 2 & 1+a & 2 \\ 1-a & 2 & 1+a \end{vmatrix} = (1+a)\cdot[(1+a)^2-4] - 2\cdot(2(1+a) - 2(1-a)) + (1-a)\cdot[4-\underbrace{(1+a)(1-a)}_{1-a^2}]$

$= (1+a)\cdot(a^2+2a-3) - 2(4a) + (1-a)\cdot(3+a^2)$

$$= \underline{a^2} \underline{-12a} \underline{-3} + \cancel{a^3} \underline{+2a^2} \underline{-3a} \underline{-8a} \underline{+3} \underline{+a^2} \underline{-3a} - \cancel{a^3}$$

$$= 4a^2 - 12a = \underline{4a(a-3)}$$

$|A| = 0$ for $a=0,\ a=3$
$|A| \neq 0$ for $a \neq 0, 3$ $\Big\}$ En løsn. for $\underline{\underline{a \neq 0,3}}$

d) $|A| = 0 \iff$ ingen eller uendelig mange løsn.

$\underline{a=0},\ \underline{a=3}$

$\underline{a=0}$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 1 & 2 & 0 \\ 1 & 2 & 1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\underline{\text{uend. mange løsn.}}$

$\underline{a=3}$:

$$\left(\begin{array}{ccc|c} 4 & 2 & -2 & 6 \\ 2 & 4 & 2 & 9 \\ -2 & 2 & 4 & 0 \end{array}\right) \cdot \tfrac{1}{2} \to \left(\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 2 & 4 & 2 & 9 \\ -2 & 2 & 4 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & 3 & 3 & 6 \\ 0 & 3 & 3 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & 3 & 3 & 6 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

$\underline{\text{ingen løsn.}}$

Ingen løsn for $\underline{\underline{a=3}}$

Når er $|A| = 0$:

- Når A har en null-rad eller null-kolonne, så er $|A| = 0$.

Eks:
$$\begin{vmatrix} 0 & 1 & 2 \\ 0 & 7 & 3 \\ 0 & -1 & 4 \end{vmatrix} = 0$$

- Når A har to like rader eller kolonner, så er $|A| = 0$.

Eks:
$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0 \qquad \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \; -1 \;\; \downarrow \;\; \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 7 & 1 \\ 2 & 3 & 2 \\ 1 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 7 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} \; -1 = \begin{vmatrix} 1 & 2 & 1 \\ 7 & 3 & 4 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

- Når A har en rad (eller kolonne) som er et multippel av en annen rad (eller kolonne), så $|A| = 0$.

Eks:
$$\begin{vmatrix} 1 & 4 & -1 \\ 0 & 2 & 0 \\ -1 & -4 & 1 \end{vmatrix} \; 1 \rightarrow \begin{vmatrix} 1 & 4 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$