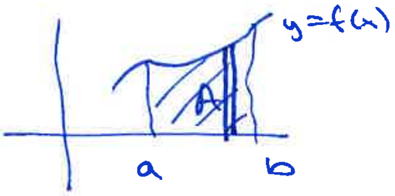
Plan

1. Økonomiske anvendelser av integrasjon
2. Partiellderivasjon

Repetisjon:

a) Areal beregning:

i)

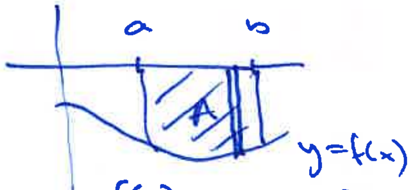


$f(x) \geq 0$ i $[a,b]$

$$A = \int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b$$
$$= F(b) - F(a)$$

der $F(x)$ er en antiderivert til $f(x)$.

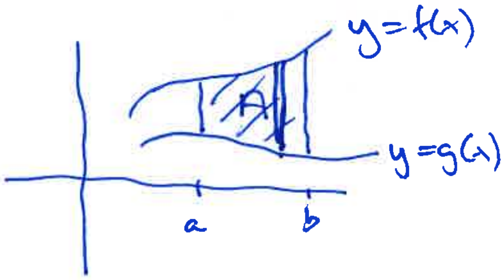
ii)



$f(x) \leq 0$ i $[a,b]$

$$A = -\int_a^b f(x)\,dx$$

iii)



$$A = \int_a^b f(x) - g(x)\,dx$$

$f(x) \geq g(x)$ i $[a,b]$

b) Uegentlige integral:

$\int_a^b f(x)\,dx$ der
- når $a/b = \pm\infty$
- når $f(a)$ eller $f(b)$ ikke er definert

⇓

Løses vha grenseverdier

Oppgave 19:

2c $\displaystyle\int_0^1 \frac{1}{x^2+5x+6}\,dx = \Big[F(x)\Big]_0^1$

$= \Big[\ln|x+2| - \ln|x+3|\Big]_0^1 = (\ln 3 - \ln 4) - (\ln 2 - \ln 3)$

$\ln \frac{|x+2|}{|x+3|}$

$= 2\ln 3 - \ln(2^2) - \ln 2$

$= \underline{\underline{2\ln 3 - 3\ln 2}}$

$\displaystyle\int \frac{1}{x^2+5x+6}\,dx = \int \frac{1}{x+2} + \frac{-1}{x+3}\,dx$

$= \ln|x+2| - \ln|x+3| + C$

$x^2+5x+6 = (x+2)(x+3)$

$\displaystyle\frac{1}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3} \quad | \cdot FN$

$1 = A(x+3) + B(x+2)$

$1 = (A+B)x + (3A+2B)$

$A+B=0 \qquad B=-A$

$3A+2B=1 \qquad 3A-2A=1$

$\underline{A=1} \quad \underline{B=-1}$

d) $\displaystyle\int_0^1 \frac{1}{x^2+4x+4}\,dx = \Big[-\frac{1}{x+2}\Big]_0^1 = \left(-\frac{1}{3}\right) - \left(-\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{3} = \underline{\underline{\frac{1}{6}}}$

$\displaystyle\frac{1}{x^2+4x+4} = \frac{A}{x+2} + \frac{B}{(x+2)^2} \quad | \cdot FN \quad \rightarrow \quad 1 = A(x+2) + B$

$1 = Ax + (2A+B)$

$x^2+4x+4 = (x+2)^2$

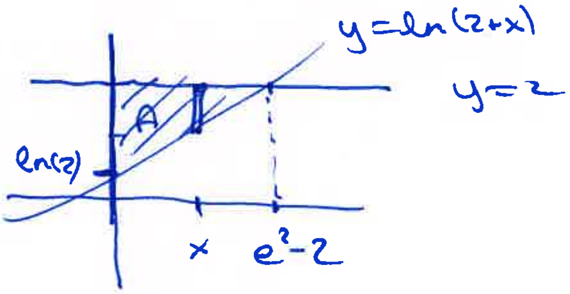
$\underline{A=0} \quad \underline{B=1}$

$\displaystyle\int \frac{1}{x^2+4x+4}\,dx = \int \frac{1}{(x+2)^2}\,dx = \int \frac{1}{u^2}\,du = \frac{1}{-1}u^{-1} + C = -\frac{1}{x+2} + C$

$u = x+2$

$du = 1 \cdot dx$

5. Området begrenset av: $\begin{cases} \text{Grafen } y = \ln(2+x) \\ \text{Linjen } y = 2 \\ y\text{-aksen} \end{cases}$



$\underline{y=2}$: $\ln(2+x) = 2$

$2+x = e^2$

$x = \underline{e^2-2}$

$\underline{x=0}$: $y = \ln(2)$

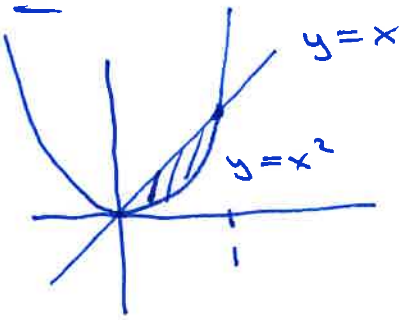
$$A = \int_0^{e^2-2} 2 - \ln(2+x)\,dx = \quad \boxed{\begin{aligned} u &= 2+x \\ du &= 1\cdot dx \end{aligned}} \quad \begin{aligned} x=0 &\to u=2 \\ x=e^2-2 &\to u=e^2 \end{aligned}$$

$$= \int_2^{e^2} 2 - \ln(u)\,du = \Big[2u - (u\cdot\ln u - u)\Big]_2^{e^2}$$

$$= \Big[3u - u\ln u\Big]_2^{e^2} = \left(3e^2 - e^2\cdot\ln(e^2)\right) - \left(6 - 2\ln 2\right)$$

$$= 3e^2 - 2e^2 - 6 + 2\ln 2 = \underline{\underline{e^2 - 6 + 2\ln 2}}$$

6. Området begrenset av $\begin{cases} y = x^2 \\ y = x \end{cases}$



$\underline{\text{Skjæringspkt}}$:

$x^2 = x$

$x^2 - x = 0$

$x\cdot(x-1) = 0$

$\underline{x=0}$, $\underline{x=1}$

$$A = \int_0^1 x - x^2\,dx = \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1$$

$$= \left(\tfrac{1}{2} - \tfrac{1}{3}\right) - (0) = \underline{\underline{1/6}}$$

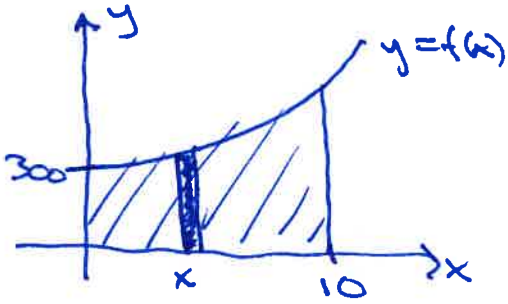
1 Økonomiske anvendelser av integrasjon

i) Kontantstrømmer
ii) Statistikk
iii) Konsument- / produsentoverskudd

} regne ut summer av størrelser som endrer seg kontinuerlig

i) Kontinuerlige kontantstrømmer

Eks:



Leie:
300 MNOK/år

$f(x) = 300 \cdot 1{,}06^x$
(leie : MNOK/år)

Samlet leie de 10 neste årene:

Arealet under grafen til $f(x)$: $[0,10]$

$$\int_0^{10} f(x)\,dx = \int_0^{10} 300 \cdot 1{,}06^x\,dx = \left[300 \cdot \frac{1{,}06^x}{\ln(1{,}06)}\right]_0^{10}$$

$$= \left(\frac{300}{\ln(1{,}06)}\,1{,}06^{10}\right) - \left(\frac{300}{\ln(1{,}06)} \cdot 1{,}06^0\right) = \frac{300 \cdot (1{,}06^{10} - 1)}{\ln(1{,}06)}$$

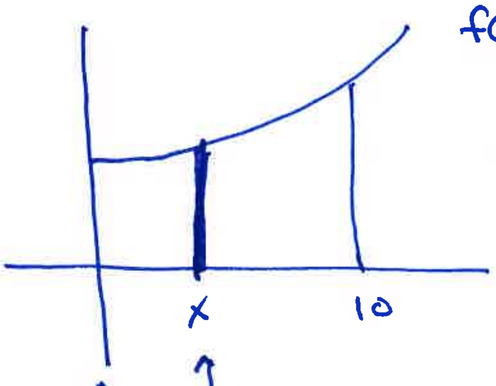
$$\simeq 4.072 \text{ MNOK}$$

$$\int a^x\,dx = \frac{a^x}{\ln(a)} + C \quad \text{eller} \quad 1.06^x = \left(e^{\ln(1{,}06)}\right)^x = e^{\ln(1{,}06)\cdot x}$$

Nåverdi: Kontinuerlig forrentning

Diskonteringsrente: $r = 10\%$

$$f(x) = 300 \cdot 1{,}06^x$$



Samlet kontantstrøm

Kontantstrøm: $f(x)\cdot dx \longrightarrow \int_0^{10} f(x)\,dx$

Nåverdi: $\dfrac{f(x)\cdot dx}{e^{0.10x}} \longrightarrow \int_0^{10} f(x)\cdot e^{-rx}\,dx$

Samlet nåverdi

$\overbrace{f(x)\cdot dx}$

$$\frac{f(x)\,dx}{e^{0.10x}}$$

Samlet nåverdi: $\displaystyle\int_0^{10} 300\cdot 1.06^x \cdot e^{-0.10x}\,dx$

$$= \int_0^{10} 300\, e^{\ln(1{,}06)x} \cdot e^{-0.10x}\,dx = \int_0^{10} 300 \cdot e^{\ln(1{,}06)x - 0.10x}\,dx$$

$$\boxed{\begin{aligned} u &= \ln(1{,}06)x - 0.10x \\ du &= (\ln(1{,}06) - 0.10)\,dx \end{aligned}}$$

$$= \int_*^* 300\, e^u \, \frac{du}{\ln(1{,}06) - 0{,}10}$$

$$= \left[\frac{300}{\ln(1{,}06) - 0{,}10}\, e^u\right]_*^*$$

$$= \left[\frac{300}{\ln(1{,}06) - 0{,}10}\, e^{\ln 1{,}06x - 0.10x}\right]_0^{10} = \frac{300}{\ln(1{,}06) - 0{,}10}\cdot\left(e^{10\ln 1{,}06 - 1} - 1\right)$$

$\simeq$

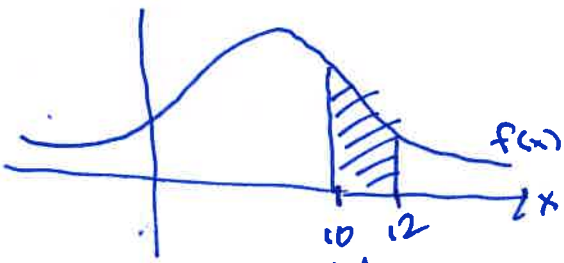
Formler:

$$\boxed{\begin{aligned} &\int_a^b f(x)\,dx = \text{samlet kontantstrøm} \\ &\int_a^b f(x)\, e^{-rx}\,dx = \text{samlet nåverdi} \end{aligned}}$$

$f(x)$: kontantstrøm raten ved ~~...~~ kontinuerlige kontantstrømmer

r : diskonteringsrente (kontinuerlig diskontering)

ii) Sannsynligheter:



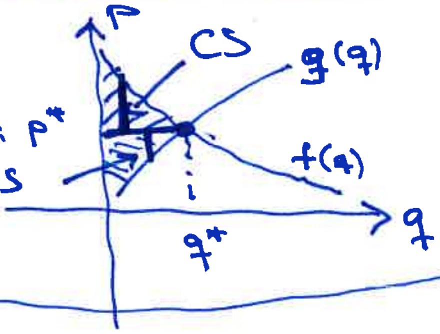
normal fordeling

$$p(10 \leq x \leq 12) = \int_{10}^{12} f(x)\,dx$$

iii) Konsument/produsentoverskudd:

p : pris

q : kvantum



$p = f(q)$ omvendt etterspørselsfn.

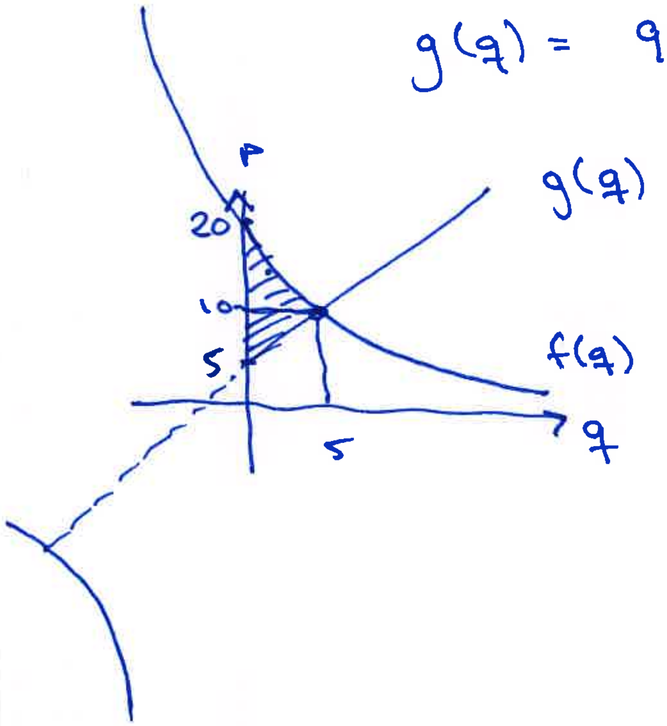
$p = g(q)$ omvendt tilbudsfunksjon

$$\boxed{\begin{aligned} CS &= \text{konsumentoverskudd} = \int_0^{q^*} f(q) - p^*\,dq \\ PS &= \text{produsentoverskudd} = \int_0^{q^*} p^* - g(q)\,dq \end{aligned}}$$

Eks:

$f(q) = \frac{100}{q+5}$ omvendt ettersp. fn.

$g(q) = q+5$ " tilbuds fn.



$\frac{100}{q+5} = q+5$

$100 = (q+5)^2$

$q+5 = \pm\sqrt{100} = \pm 10$

$q = -5 \pm 10$

$q_1 = -15 \quad q_2 = 5$

$q^* = 5$

$p^* = 10$

$CS = \int_0^5 \frac{100}{q+5} - 10 \, dq$

$= \left[100 \ln|q+5| - 10q \right]_0^5$

$= (100 \ln(10) - 50) - (100 \ln 5) = 100\ln(10) - 100\ln 5 - 50$

$= 100(\ln 2 + \cancel{\ln 5}) - \cancel{100\ln 5} - 50 \qquad \ln 10 = \ln 2 + \ln 5$

$= 100 \ln 2 - 50 \simeq 13.9$

$PS = \frac{25}{2} = 12.5$

$PS = \int_0^5 10 - (q+5) \, dq = \int_0^5 5 - q \, dq$

$\left[5q - \tfrac{1}{2}q^2 \right]_0^5 = \left(25 - \tfrac{25}{2}\right) - 0$

$= 12.5$

2 Partiell derivasjon

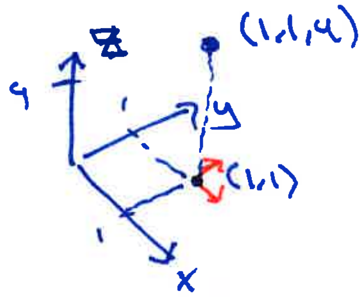
$f(x,y)$: funksjon i to variabler

Eks: $f(x,y) = 2x - 3y + 1$

$f(x,y) = x^2 - y^2 + 2y$

$f(x,y) = e^{x-y}$

Graf: $z = f(x,y)$



$z = f(1,1) = 4$

Partiellderiverte:

$$f'_x = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \longleftarrow \text{ y holdes konstant}$$

$$f'_y = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} \longleftarrow \text{ x holdes konst.}$$

Eks: $f(x,y) = 2x - 3y + 1$ $\qquad f'_x = 2$, $\quad f'_y = -3$

$f(x,y) = x^2 - y^2 + 2y$ $\qquad f'_x = 2x$, $\quad f'_y = -2y + 2$

$f(x,y) = 3xy$ $\qquad f'_x = (3xy)'_x = 3y\,(x)'_x = 3y$

$f'_y = (3xy)'_y = 3x \cdot 1 = 3x$