Plan

1 Bestemte integral og antiderivasjon
2 Beregning av areal mellom grafer
3 Uegentlige integral

Repetisjon:

Integrasjonsteknikk: i) Substitusjon, ii) Delvis integrasjon, iii) Delbrøksoppspalting

i) Substitusjon: $\int f(x)\,dx = \dots = \int g(u)\,du$

$$\boxed{\begin{aligned} u &= u(x) \\ du &= u'(x)\cdot dx \end{aligned}}$$

ii) Delvis integrasjon: $\int u' \cdot v\,dx = u\cdot v - \int u \cdot v'\,dx$

$$\boxed{\begin{aligned} u &= \dots & v &= v(x) \\ u' &= \dots & v' &= v'(x) \end{aligned}}$$

iii) Delbrøksoppspalting: $\int \frac{p(x)}{q(x)}\,dx$ integrasjon av rasjonale uttrykk.

Husk: bruk polynomdiv. først dersom graden til $p(x)$ er lik eller større enn graden til $q(x)$

Hvis graden til $q(x)$ er 1:

$$\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C$$

$$\frac{p(x)}{ax^2+bx+c} = \frac{A}{x-x_1} + \frac{B}{x-x_2}$$

Husk: Eks: $\frac{2}{x^2-4x+4} = \frac{2}{(x-2)\cdot(x-2)} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2}$

Oppgave ark 18:

7. $\int 2x^3 e^{-x^2}\,dx = \int 2x^3 \cdot e^u \cdot \frac{du}{-2x}$

$\boxed{\begin{array}{l} u=-x^2 \\ du=-2x\,dx \end{array}}$

$= \int -x^2 e^u\,du = \int u e^u\,du = ue^u - \int e^u \cdot 1\,du$

$\boxed{\begin{array}{ll} w = e^u & v = u \\ w' = e^u & v' = 1 \end{array}}$

$= ue^u - \int e^u du = ue^u - e^u + C = \underline{-x^2 e^{-x^2} - e^{-x^2} + C}$

8. $\int \sqrt{x}\, e^{\sqrt{x}}\,dx = \int \sqrt{x}\, e^u\, 2\sqrt{x}\,du$

$\boxed{\begin{array}{l} u=\sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\,dx \end{array}}$

$= \int 2u^2 e^u\,du = 2u^2 \cdot e^u - \int 4u e^u\,du$

$\boxed{\begin{array}{ll} e^u & 2u^2 \\ e^u & 4u \end{array}}$

$= 2u^2 e^u - 4(ue^u - e^u) + C$

$= 2u^2 e^u - 4ue^u + 4e^u + C = \underline{2x e^{\sqrt{x}} - 4\sqrt{x} e^{\sqrt{x}} + 4e^{\sqrt{x}} + C}$

9. $$\int \frac{\sqrt{x}+1}{1-\sqrt{x}}\,dx = \int \frac{\sqrt{x}+1}{u}(-2\sqrt{x})\,du$$

$$\boxed{\begin{array}{l} u = 1-\sqrt{x} \\ du = -\frac{1}{2\sqrt{x}}\,dx \end{array}}$$

$$\sqrt{x} = 1-u$$

$$= \int \frac{(2-u)(-2(1-u))}{u}\,du = \int \frac{(2-u)(2u-2)}{u}\,du$$

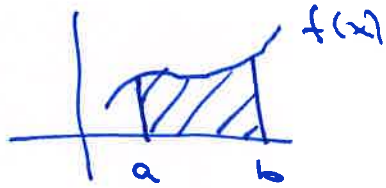
$$= \int \frac{-2u^2+6u-4}{u}\,du = \int -2u+6-\frac{4}{u}\,du$$

$$= -u^2+6u-4\ln|u|+C$$

$$= \underline{\underline{-(1-\sqrt{x})^2+6(1-\sqrt{x})-4\ln|1-\sqrt{x}|+C}}$$

① Bestemt integral:

Defn I: $\int_a^b f(x)\,dx =$ ~~areal av grafen under f(x)~~

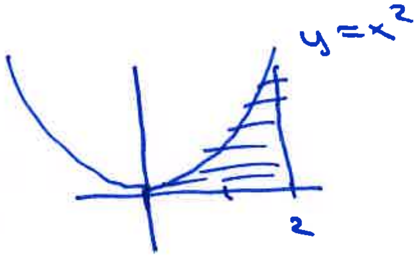


$$= \left\{ \begin{array}{l} \text{areal av område} \\ \text{under grafen til } f \\ \text{i } [a,b] \end{array} \right\}$$

$f(x) \geq 0$ for $a \leq x \leq b$

Defn 2: $\int_a^b f(x)\,dx = F(b) - F(a)$, der $F(x)$ er en antiderivert til $f(x)$.

Eks: $\int_0^2 x^2\,dx$



$$= \left[\tfrac{1}{3}x^3 + C \right]_0^2$$

Defn 2:

$$= \left(\tfrac{1}{3}\cdot 2^3 + \cancel{C}\right) - \left(\tfrac{1}{3}\cdot 0^3 + \cancel{C}\right)$$

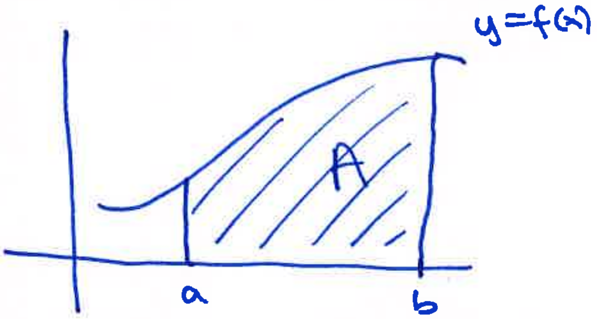
$$= \frac{8}{3} - 0 = 8/3 \simeq \underline{\underline{2.67}}$$

Teorem:

Hvis $f(x)$ er en kontinuerlig funksjon på $[a,b]$ og $f(x) \geq 0$ for $a \leq x \leq b$, så er

$$\left\{ \begin{array}{l} \text{Arealet under grafen} \\ \text{til } f(x) \text{ i } [a,b] \end{array} \right\} = F(b) - F(a)$$

(der $F(x)$ er en antiderivert til $f(x)$)



Skal forklare hvorfor

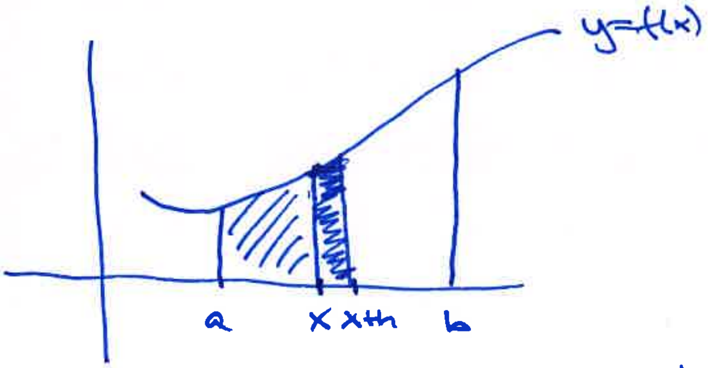
$$F(b) - F(a) = A$$

Påstand:

$A(b) = A$ vok
$A(a) = 0$ vok.
$A'(x) = f(x)$

$\Downarrow$

$F = A$: $A(b) - A(a) = A - 0 = A$.



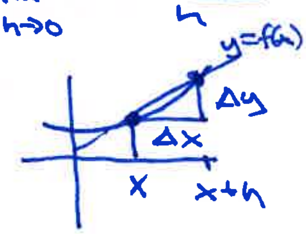
$$A(x) = \left\{ \begin{array}{l} \text{areal under grafen til} \\ f(x) \text{ i } [a,x] \end{array} \right\}$$

$$A'(x) \underset{h \text{ liten}}{\simeq} \frac{A(x+h) - A(x)}{h} = \frac{}{h}$$

$$\simeq \frac{f(x) \cdot \cancel{h}}{\cancel{h}} = f(x)$$

$f'(x) =$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



Fra nå av:
$$\int_a^b f(x)\,dx = F(b)-F(a)$$
der $F(x)$ er en anti-derivert til $f(x)$.

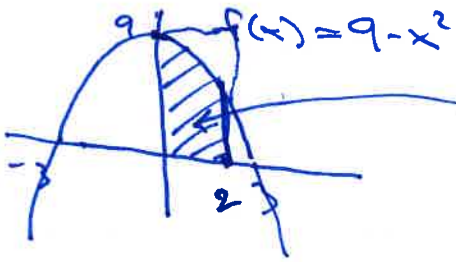
bestemt integral

(generelle defn. av bestemte integral)

Resultat: Hvis $f(x)\geq 0$ i $[a,b]$, så er
$$\int_a^b f(x)\,dx = \left\{\begin{array}{l}\text{arealet under grafen} \\ \text{til } f \text{ i } [a,b]\end{array}\right\}$$

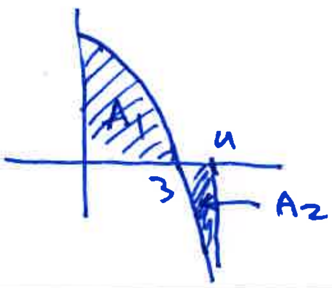
2 Arealberegning:

Eks:
$$\int_0^2 9-x^2\,dx = \left[9x-\tfrac{1}{3}x^3\right]_0^2$$
$$= \left(9\cdot 2-\tfrac{1}{3}\cdot 2^3\right)-(0) = 18-\tfrac{8}{3} = \tfrac{46}{3} \simeq \underline{\underline{15{,}3}}$$



Areal $\simeq \underline{\underline{15{,}3}}$

Eks:
$$\int_0^4 9-x^2\,dx = \left[9x-\tfrac{1}{3}x^3\right]_0^4 = \left(9\cdot 4-\tfrac{4^3}{3}\right)-0$$
$$= 36-\tfrac{64}{3} = \tfrac{44}{3} \simeq \underline{14{,}7}$$



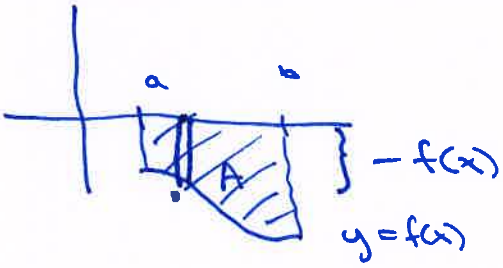
$$\int_0^4 f(x)\,dx = \underbrace{\int_0^3 f(x)\,dx}_{A_1} + \underbrace{\int_3^4 f(x)\,dx}_{-A_2}$$

Hvis $f(x) \le 0$ i $[a,b]$:

$$\int_a^b f(x)\,dx = -A$$

$$\Updownarrow$$

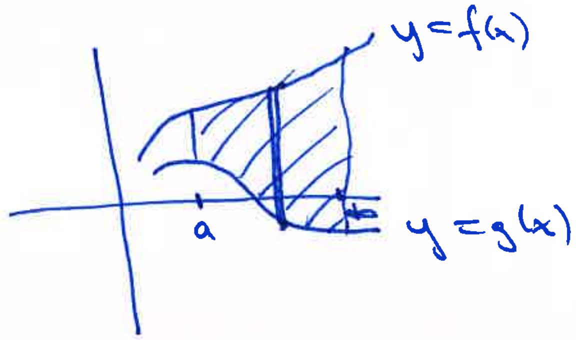
$$A = \int_a^b -f(x)\,dx$$



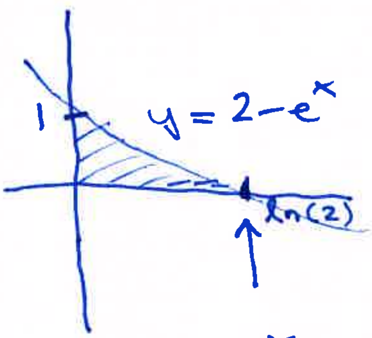
Mer generelt: Arealet mellom to grafer

Når $f(x) \ge g(x)$ i $[a,b]$, er arealet av området mellom grafene til f og g i $[a,b]$ lik:

$$\boxed{A = \int_a^b f(x) - g(x)\,dx}$$



Eks: Finn arealet av området begrenset av x-aksen, y-aksen og grafen til $f(x) = 2 - e^x$



$2 - e^x = 0$
$e^x = 2$
$x = \ln 2$

$$A = \int_0^{\ln(2)} 2 - e^x\,dx$$

$$= \Big[2x - e^x\Big]_0^{\ln(2)}$$

$$= \left(2\ln(2) - e^{\ln(2)}\right) - \left(0 - e^0\right)$$

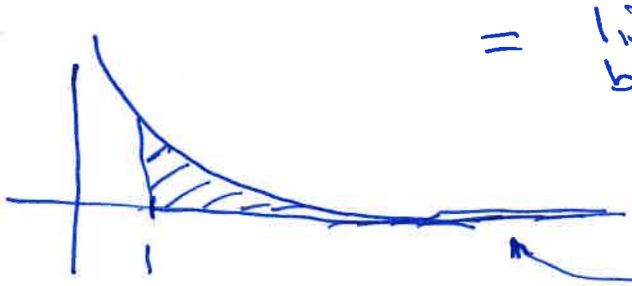
$$= 2\ln 2 - 2 + 1 = \underline{\underline{2\ln 2 - 1}} \simeq 0.4$$

$$\boxed{\text{Husk: } \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$

3 Uegentlige integral

Eks: $\int_1^\infty \frac{1}{x}\,dx = \left[\ln|x|\right]_1^\infty = \cancel{\ln\infty - \ln 1}$

$= \lim_{b\to\infty}(\ln b - \ln 1) = \infty$

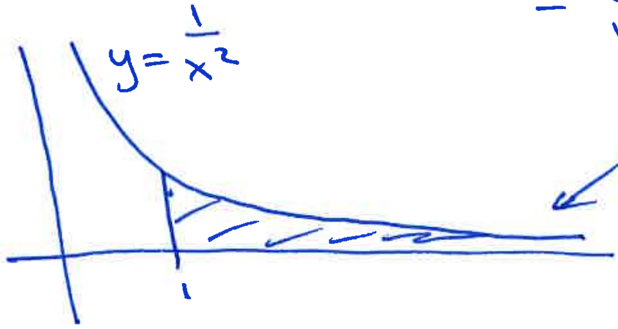


arealet er uendelig stort

Eks: $\int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^\infty = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b$

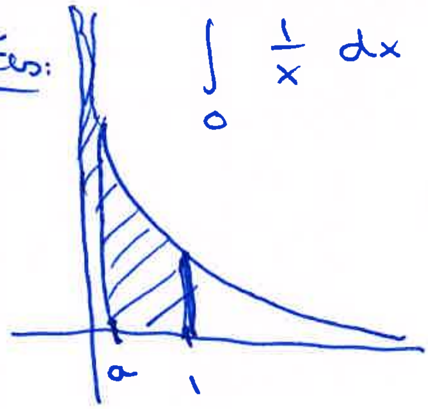
$= \lim_{b\to\infty}\left(\left(-\frac{1}{b}\right) - \left(-\frac{1}{1}\right)\right)$

$= \lim_{b\to\infty}\left(-\frac{1}{b} + 1\right) = \underline{\underline{1}}$



arealet er 1

Eks:



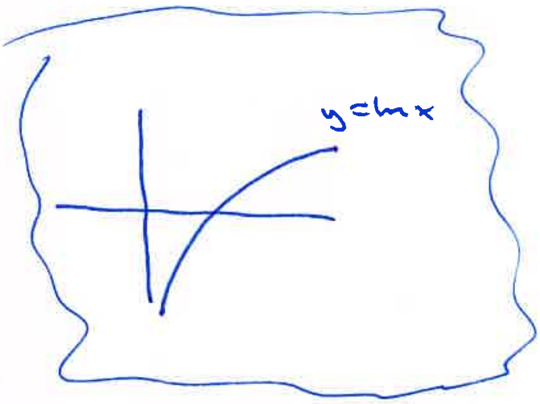
$$\int_0^1 \frac{1}{x}\,dx = \left[\ln|x|\right]_0^1 \text{"}=\text{"} \ln 1 - \ln 0 = 0 - (-\infty) = \infty\text{"}$$

$$\lim_{a\to 0}\left[\ln|x|\right]_a^1 = \lim_{a\to 0}\left[\ln|x|\right]_a^1$$

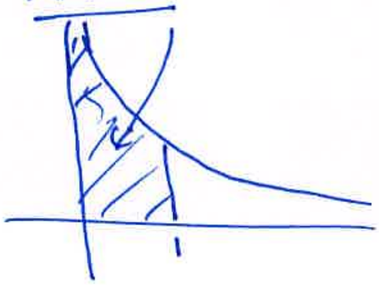
$$= \lim_{a\to 0}(\ln 1 - \ln a)$$

$$= \lim_{a\to 0}(0 - \ln a)$$

$$= -(-\infty) = \underline{\underline{\infty}}$$



Arealet $= \infty$



Vi får uegentlig integral $\int_a^b f(x)\,dx$ når

i) $a = -\infty$ eller $b = \infty$

ii) Hvis $f(x)$ ikke er definert i hele $[a,b]$

Eks: $\int \frac{x^2}{4-x^2}\,dx = \int -1 + \frac{4}{4-x^2}\,dx$

$= -x + \int \frac{4}{4-x^2}\,dx$

$\boxed{\begin{array}{l} \text{Polynomdiv:} \\ x^2 : (-x^2+4) = -1 + \frac{4}{4-x^2} \\ \underline{-(x^2-4)} \\ \quad 4 \end{array}}$

Delbrøksoppspaltning:

$\frac{4}{4-x^2}$:

$(4-x^2) = (2+x)(2-x)$

$\left(= -(x+2)(x-2) \right)$

$\frac{4}{4-x^2} = \frac{A}{2+x} + \frac{B}{2-x} \quad | \cdot FN$

$4 = A\cdot(2-x) + B(2+x)$

$\boxed{\begin{array}{ll} x=2: & 4 = 4B \quad \underline{B=1} \\ x=-2: & 4 = 4A \quad \underline{A=1} \end{array}}$

$\boxed{\begin{array}{l} 4 = (2A+2B) + (-A+B)x \\ 2A+2B = 4 \qquad 4A=4 \Rightarrow A=1 \\ -A+B=0 \Rightarrow B=A \\ \underline{A=B=1} \end{array}}$

$\cancel{\frac{x^2}{4-x^2} = \frac{A}{2-x} + \frac{B}{2+x} \quad | \cdot FN}$

$\cancel{x^2 = A(2+x) + B(2-x)}$

$\cancel{x=-2: \quad 4 = 4B \quad B=1}$

$\cancel{x=2: \quad 4 = 4A \quad A=1}$

$x^2 = (2A+2B) + (A-B)x$

$$\int \frac{x^2}{4-x^2}\,dx = \int -1 + \frac{4}{4-x^2}\,dx$$

$$= -x + \int \frac{1}{2-x} + \frac{1}{2+x}\,dx$$

$$= -x + (-\ln|2-x|) + \ln|2+x| + C$$

$$= -x + \ln|2+x| - \ln|2-x| + C$$

$$= -x + \ln\frac{|2+x|}{|2-x|} + C$$

$$\int \frac{1}{2-x}\,dx = \int \frac{1}{u}\cdot\frac{du}{-1}$$

$$\boxed{\begin{array}{l} u = 2-x \\ du = -1\cdot dx \end{array}}$$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= -\ln|2-x| + C$$

$$\int_3^\infty \frac{x^2}{4-x^2}\,dx = \Big[-x + \ln\left|\frac{2+x}{2-x}\right|\Big]_3^\infty$$

$$= (-\infty + 0) - (-3 + \ln 5)$$

$$= -\infty$$

$$\lim_{x\to\infty} \frac{2+x}{2-x} = \lim_{x\to\infty} \frac{1}{-1} = -1$$