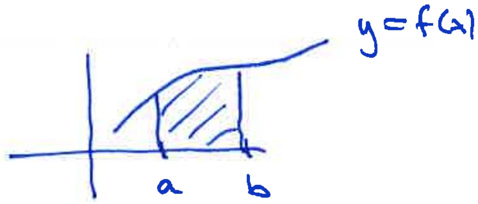
Plan

1. Substitusjon
2. Delvis integrasjon
3. Integrasjon av rasjonale uttrykk og delbrøksoppspaltning

Repetisjon:

Bestemt integral:

$\int_a^b f(x)\,dx$ = areal under grafen til f i $[a,b]$



$f(x) \geq 0$ i $[a,b]$
f kont.

Ubestemt integral:

$\int f(x)\,dx = F(x) + C$ hvis $F'(x) = f(x)$, dvs at $F(x)$ er antiderivert til $f(x)$

integrasjonskonst → gir den generelle antiderivérte

Integrasjonsregler:

i) $\int x^n\,dx = \frac{1}{n+1} \cdot x^{n+1} + C \quad (n \neq -1)$

ii) $\int 1/x\,dx = \ln|x| + C$

iii) $\int u \pm v\,dx = \int u\,dx \pm \int v\,dx$

iv) $\int c \cdot u\,dx = c \cdot \int u\,dx$

u, v: uttrykk i x
c: konstant

v) $\int e^x\,dx = e^x + C$

$\int a^x\,dx = \frac{1}{\ln(a)} \cdot a^x + C \quad (a > 0)$

Oppgavearh 17:

8d) $\int 3^x dx = \underline{\frac{1}{\ln(3)} \cdot 3^x + C}$ ← bruker $\int a^x dx$ for $a=3$

Alt: $y = 3^x$

$\ln y = \ln(3^x) = x \cdot \ln(3) \quad | \, e^*$

$y = e^{x \cdot \ln(3)} = e^{\ln(3) \cdot x}$

$$\int 3^x dx = \int e^{\ln(3) \cdot x} dx = \frac{1}{\ln(3)} \cdot e^{\ln(3) \cdot x} + C$$

$$= \underline{\frac{1}{\ln(3)} \cdot 3^x + C}$$

Ex: $\int \frac{x}{1-x} dx = \int \frac{x}{u} \cdot \frac{du}{-1} = \frac{1}{-1} \cdot \int \frac{1-u}{u} du$

$\boxed{\begin{array}{c} u = 1-x \\ du = -1 \cdot dx \end{array}}$ $\quad x = 1-u$

$du = u' \cdot dx$ "Omvendt kjerneregel"

$dx = \frac{du}{-1}$

$$- \int \left(\frac{1}{u} - 1\right) du = -\left(\ln|u| - u\right) + C$$

$$= \underline{\underline{-\ln|1-x| + 1 - x + C}}$$

① Substitusjon

$$\int f(x)\,dx = \dots = \int g(u)\,du$$

$$\boxed{\begin{aligned} u &= u(x) \\ du &= u'(x)\cdot dx \end{aligned}} \longleftarrow \text{et uttrykk i } x$$

Eks:

$$\int \frac{2x}{1-x^2}\,dx = \int \frac{2x}{u}\,\frac{du}{-2x} = \int \frac{\cancel{2x}}{u}\cdot\frac{1}{-\cancel{2x}}\,du$$

$$\boxed{\begin{aligned} u &= 1-x^2 \\ du &= -2x\cdot dx \end{aligned}}$$

$$= \int -\frac{1}{u}\,du = -\ln|u| + C = \underline{\underline{-\ln|1-x^2| + C}}$$

Eks:

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{\cancel{x}}{\sqrt{u}}\,\frac{du}{2\cancel{x}} = \int \frac{1}{2\sqrt{u}}\,du$$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \end{aligned}}$$

$$= \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\left(\frac{1}{1/2}u^{1/2}\right) + C$$

$$= u^{1/2} + C = \sqrt{u} + C = \underline{\underline{\sqrt{x^2+1} + C}}$$

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{\cancel{x}}{u}\cdot\frac{\sqrt{x^2+1}}{\cancel{x}}\,du$$

$$\boxed{\begin{aligned} u &= \sqrt{x^2+1} \\ u' &= \underbrace{\frac{1}{2\sqrt{x^2+1}}\cdot 2x}_{u'} \end{aligned}}$$

$$= \int \frac{1}{\cancel{u}}\cdot\frac{\cancel{u}}{1}\,du$$

$$= \int 1\,du = u + C$$

$$= \underline{\underline{\sqrt{x^2+1} + C}}$$

$$du = u'\cdot dx$$

$$du = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}\,dx \Rightarrow dx = \frac{\sqrt{x^2+1}}{x}\,du$$

Når bruker vi substitusjon:

* $\int f(x)\,dx$ der hele $f(x)$ eller deler av $f(x)$ har en naturlig kjerne

* Når andre metoder ikke fungerer.

Eks: Oppgaveark 17, opps. 10: $\int \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx$

Subst: $u = 1-\sqrt{x}$

② Delvis integrasjon: "omvendt produktregel"

Ex: $\cancel{\int x \cdot \ln(x)\,dx \neq \frac{1}{2}x^2 \cdot \int \ln x\,dx}$

$$\left.\begin{aligned} (uv)' &= u'v + uv' \\ \int (uv)'\,dx &= \int u'v + uv'\,dx \\ u \cdot v &= \int u'v\,dx + \int uv'\,dx \end{aligned}\right\}$$

$$\boxed{\int u' \cdot v\,dx = u \cdot v - \int uv'\,dx}$$

formel for delvis integrasjon

Eks: $\int x \cdot e^x\,dx = \cancel{\frac{1}{2}x^2 e^x - \int \frac{1}{2}x^2 e^x\,dx}$

$\cancel{\boxed{\begin{aligned} u &= \tfrac{1}{2}x^2 & v &= e^x \\ u' &= x & v' &= e^x \end{aligned}}}$

$\int x \cdot e^x\,dx = \int e^x \cdot x\,dx = xe^x - \int e^x \cdot 1\,dx$

$\boxed{\begin{aligned} u &= e^x & v &= x \\ u' &= e^x & v' &= 1 \end{aligned}}$

$= xe^x - \int e^x\,dx$

$= xe^x - e^x + C$

$$\int \ln(x)\,dx = \int 1\cdot\ln(x)\,dx = x\cdot\ln(x) - \int \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx$$

$$\boxed{\begin{array}{ll} u = x & v = \ln(x) \\ u' = 1 & v' = 1/x \end{array}}$$

$$= x\ln(x) - \int 1\,dx = \underline{\underline{x\ln x - x + C}}$$

$$\boxed{\int \ln x\,dx = x\ln x - x + C}$$

Integrasjonsregel for $\ln(x)$.

Eks: $\int x^2\cdot\ln(x)\,dx =$

$$\boxed{\begin{array}{ll} u = \frac{1}{3}x^3 & v = \ln(x) \\ u' = x^2 & v' = 1/x \end{array}}$$

$$= \frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^3\cdot\frac{1}{x}\,dx$$

$$= \frac{1}{3}x^3\ln(x) - \frac{1}{3}\int x^2\,dx = \frac{1}{3}x^3\ln(x) - \frac{1}{3}\cdot\frac{1}{3}x^3 + C$$

$$= \underline{\underline{\frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3 + C}}$$

Eks: $\int \frac{\ln x}{x}\,dx =$

Substitusjon: $\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}}$

$$\int \frac{u}{\cancel{x}}\cdot\cancel{x}\,du = \int u\,du$$

$$= \frac{1}{2}u^2 + C$$

$$= \underline{\underline{\frac{1}{2}(\ln x)^2 + C}}$$

$$\int \frac{\ln x}{x}\,dx = \int \frac{1}{x}\cdot \ln x\,dx$$

$$\boxed{\begin{matrix} u = \ln x & v = \ln x \\ u' = 1/x & v' = 1/x \end{matrix}}$$

$$= (\ln x)^2 - \int \ln x \cdot \frac{1}{x}\,dx$$

$$\int \frac{\ln x}{x}\,dx = (\ln x)^2 - \int \frac{\ln x}{x}\,dx$$

$$2\int \frac{\ln x}{x}\,dx = (\ln x)^2$$

$$\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2} + C$$

$$F(x) + C_1 = \ln(x)^2 \div (F(x) + C_2)$$

$$2F(x) = \ln(x)^2 - C_2 - C_1$$

③ Integrasjon av rasjonale uttrykk og delbrøksoppspaltning.

Rasjonale uttrykk: $\frac{p(x)}{q(x)}$ der p, q er polynomer

Eks: $\int \frac{2x}{1-x}\,dx$ $\int \frac{x}{x^2-5x+6}\,dx$

a) Hvis $\deg(p) \geq \deg(q)$: Bruk polynomdivisjon først.

$\boxed{\text{graden til teller} \geq \text{graden til nevner}}$

$$\int \frac{2x}{1-x}\,dx = \int -2 + \frac{2}{1-x}\,dx$$

$$2x : (-x+1) = \underline{-2} \quad \text{kvotient}$$
$$\underline{-(2x-2)}$$
$$2 \text{ rest}$$

$$\frac{2x}{1-x} = -2 + \frac{2}{1-x}$$

$$= -2x + \int \frac{2}{1-x}\,dx$$

$\boxed{u = 1-x \quad du = -1 \cdot dx}$

$$= -2x + \int \frac{2}{u}\,\frac{du}{-1}$$

$$= -2x - 2\int \frac{1}{u}\,du$$

$$= -2x - 2\ln|u| + C$$

$$= -2x - 2\ln|1-x| + C$$

b) Graden til nevner er 1:

$$\int \frac{A}{ax+b}\,dx = \int \frac{A}{u}\,\frac{du}{a}$$

$$\boxed{\begin{array}{l} u = ax+b \\ du = a \cdot dx \end{array}}$$

$$= \frac{A}{a}\int \frac{1}{u}\,du = \frac{A}{a}\ln|u| + C$$

$$= \underline{\underline{\frac{A}{a}\ln|ax+b| + C}}$$

$$\boxed{\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C}$$

formel i tilfellet at nevner har grad 1 og teller er konstant

Eks: $\displaystyle\int \frac{2}{1-2x}\,dx = \frac{2}{-2}\ln|1-2x| + C$

$$= \underline{\underline{-\ln|1-2x| + C}}$$

c) Graden til nevner er ≥ 2 : Delbrøksoppspaltning

Eks: $\boxed{\int \frac{2}{x^2-4}\,dx}$ $\int \frac{x}{x^2-4}\,dx$

$x^2-4 = (x-2)\cdot(x+2)$

① Faktoriser nevner

$\frac{2}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$

② Skriv ned delbrøks-oppspaltning vha. A, B.

$\frac{2}{x^2-4} = \frac{1/2}{x-2} + \frac{-1/2}{x+2}$

③ Finn A og B.

$A = 1/2$ $B = -1/2$

$\frac{2}{x^2-4} = \frac{A}{(x-2)} + \frac{B}{x+2} \quad \Big| \cdot (x-2)(x+2)$

$2 = \frac{A}{\cancel{x-2}}\cdot\cancel{(x-2)}(x+2) + \frac{B}{\cancel{x+2}}\cdot(x-2)\cancel{(x+2)}$

$2 = A\cdot(x+2) + B(x-2)$

Metode I:

$x=-2$: $2 = A\cdot 0 + B\cdot(-4)$

$B = 2/-4 = -1/2$

$x=2$: $2 = A\cdot 4 + B\cdot 0$

$A = 2/4 = 1/2$

Metode 2:

$$2 = Ax + 2A + Bx - 2B$$
$$= Ax + Bx + 2A - 2B$$
$$= (A+B)x + (2A-2B)$$

$0x + 2$

$\boxed{\begin{matrix} A+B=0 \\ 2A-2B=2 \end{matrix}}$

$B = -A$

$2A - 2(-A) = 2$

$4A = 2$

$B = -1/2$

$A = 2/4 = 1/2$

$$\frac{2}{x^2-4} = \frac{1/2}{x-2} + \frac{-1/2}{x+2} \Rightarrow \int \frac{2}{x^2-4}\,dx = \int \frac{1/2}{x-2}\,dx + \int \frac{-1/2}{x+2}\,dx$$

$$= \frac{1/2}{1}\ln|x-2| + \frac{(-1/2)}{1}\ln|x+2| + C$$

$$= \frac{1}{2}\ln|x-2| - \frac{1}{2}\ln|x+2| + C$$

Eks:

$$\int \frac{x+1}{x^2-4x+4}\,dx = \int \frac{1}{x-2} + \frac{3}{(x-2)^2}\,dx$$

$x^2-4x+4 = (x-2)^2$

$x^2-4x+4=0$

$x = \frac{4 \pm \sqrt{16-4\cdot 4}}{2}$

$x_1 = 2 \quad x_2 = 2$

$$\frac{x+1}{x^2-4x+4} = \frac{A}{x-2} + \frac{B}{(x-2)^2} \quad \Big| \cdot (x-2)^2$$

$x+1 = A\cdot(x-2) + B$

$x+1 = Ax + (-2A+B)$

$A=1 \qquad -2A+B=1$

$-2+B=1$

$B=3$

$$= \frac{1}{1}\ln|x-2| + \int \frac{3}{(x-2)^2}\,dx$$

$\boxed{u = x-2 \quad du = dx}$

$$= \ln|x-2| + \int \frac{3}{u^2}\,du$$

$$= \ln|x-2| + 3\cdot\int u^{-2}\,du = \ln|x-2| + 3\cdot\frac{1}{(-1)}u^{-1} + C$$

$$= \ln|x-2| - 3\,\frac{1}{x-2} + C$$