Plan:

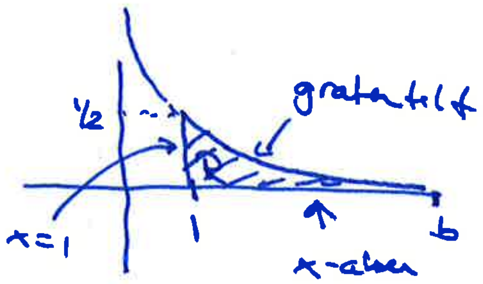
1. Funksjoner i to variable
2. Lineære funksjoner
3. Partiellderivasjon

Pensum: [7.1–7.3]

Husk: Fagoppgave **Frist:** Fredag kl. 12

Oppg 1, Oppgavesett 20:

$$f(x) = \frac{1}{x^2+x}$$



$$A(R) = \int_1^\infty f(x) - 0 \, dx = \int_1^\infty f(x)\, dx$$

$$= \lim_{b\to\infty} \int_1^b f(x)\, dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2+x}\, dx$$

1) $\int \frac{1}{x^2+x}\, dx = \ldots\ldots = F(x) + C$

2) $\int_1^b \frac{1}{x^2+x}\, dx = \Big[F(x)\Big]_1^b = F(b) - F(1) \longleftarrow$ uttrykk i b

3) $\lim_{b\to\infty} \big(F(b) - F(1)\big) = \underline{\underline{\text{arealet}}}$

Eivind Eriksen, Kontor B4-032, eivind.eriksen@bi.no
http://www.dr-eriksen.no/teaching/MET1180/
Handelshøyskolen BI
Økonomi og Ledelse

1 Funksjoner i to variable

Eks: $f(x,y) = \underbrace{x^2+y^2}_{\text{funksjonsuttrykk, et uttrykk i } x \text{ og } y}$

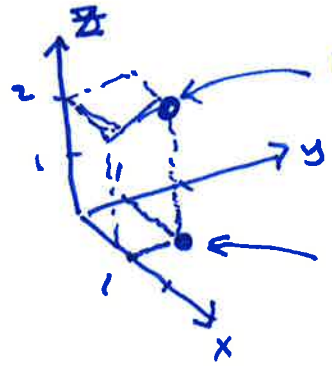
Funksjonsverdier:

(x,y)	$f(x,y) = z$
$(0,0)$	$0 = f(0,0)$
$(1,0)$	$1 = f(1,0)$
$(0,1)$	$1 = f(0,1)$
$(1,1)$	$2 = f(1,1)$
$\vdots$	$\vdots$
input	output = funksjonsverdi

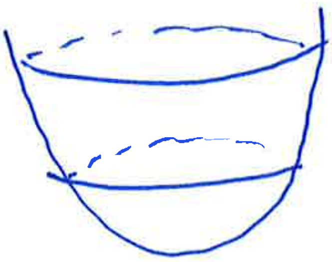
Grafen til f:

Alle tall-trepler (x,y,z), der (x,y) er et tallpar i D_f, og $z = f(x,y)$

Skrivemåte: $z = f(x,y)$



$z = f(x,y)$ = høyden over xy-planet



grafen til f

$\mathbb{R}^2$: alle tallpar (x,y) med x,y reelle tall

Definisjonsområde til f: D_f

Alle tallpar (x,y) som tillates brukt (satt inn) i f

$D_f \subseteq \mathbb{R}^2$

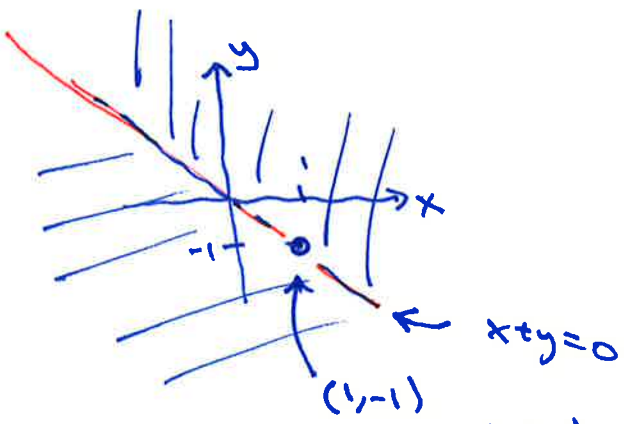
Eks: $f(x,y) = x^2+y^2$, $D_f = \mathbb{R}^2$

Eks: $f(x,y) = \dfrac{xy}{x+y}$ rasjonal funksjon

D_f: alle tall (x,y) slik at $x+y \neq 0$

$D_f = \{(x,y) \in \mathbb{R}^2 : x+y \neq 0\}$

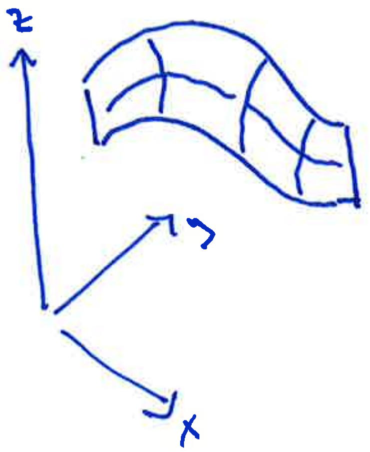
$= \{(x,y) : x+y \neq 0\}$



$x+y=0$
$y=-x$

$f(1,-1) = \dfrac{1\cdot(-1)}{1+(-1)}$ er ikke definert

Husk: Grafen til $f(x,y)$ er en flate



Snitt:

a) Nivåkurver: $\boxed{f(x,y) = c}$ ⟵ $z = f(x,y)$

Setter $z = c$
(en konstant)

Eks: $f(x,y) = x^2 + y^2$

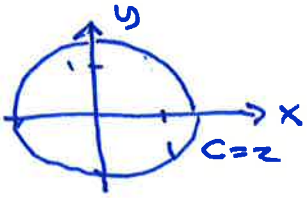
$x^2 + y^2 = c$:

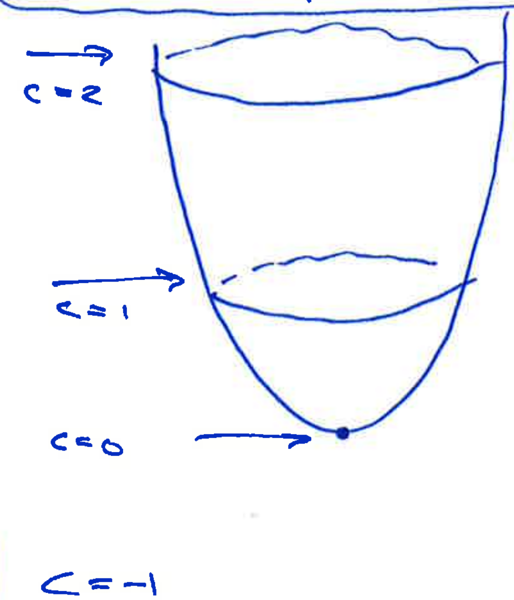
$c > 0$: sirkel m/radius $\sqrt{c}$
senter i $(0,0)$

$c = 0$: ett punkt $(0,0)$

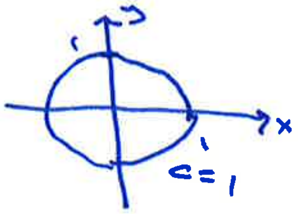
$c < 0$: ingen punkt.

$c = 2$: $x^2 + y^2 = 2$

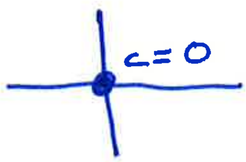




$c = 1$: $x^2 + y^2 = 1$



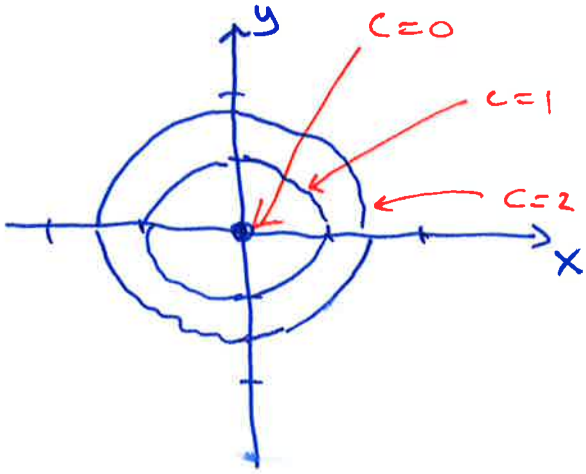
$c = 0$: $x^2 + y^2 = 0$



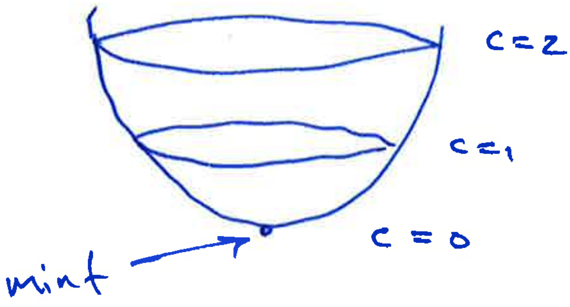
$c = -1$: $x^2 + y^2 = -1$

ingen punkt.

Vanlig framstilling:



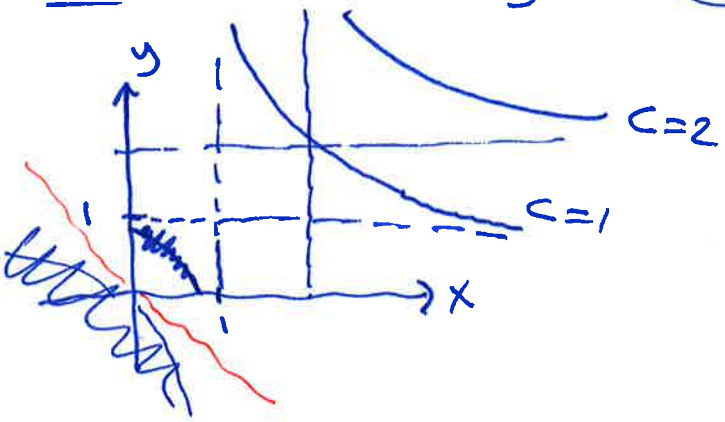
Eks: $f(x,y)=x^2+y^2$



min $f=0$

max f eksisterer ikke ($\to\infty$)

Eks: $f(x,y)=\dfrac{xy}{x+y}$, $D_f: x,y>0$



Nivåkurver: $f(x,y)=c$

c=0: $\dfrac{xy}{x+y}=0$

$xy=0$

$\cancel{x=0}$ eller $\cancel{y=0}$

ingen pkt.

C=2: $\dfrac{xy}{x+y}=2$

$xy=2(x+y)$

$xy-2y=2x$

$y(x-2)=2x$

$y=\dfrac{2x}{x-2}=\dfrac{2x-4+4}{x-2}$

$y=2+\dfrac{4}{x-2}$

c=1: $\dfrac{xy}{x+y}=1$

$xy=1\cdot(x+y)$

$xy-y=x$

$\dfrac{(x-1)\cdot y}{x-1}=\dfrac{x}{x-1}$

$y=\dfrac{x}{x-1}=\dfrac{x-1+1}{x-1}$

$y=1+\dfrac{1}{x-1}$

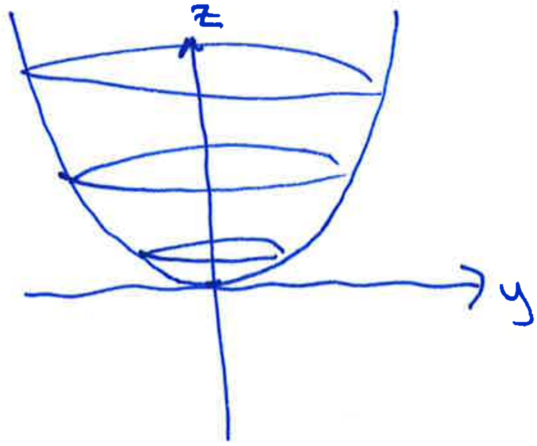
b) Snitt: $x = c$ eller $y = c$

Eks: $f(x,y) = x^2 + y^2$

x=0: $z = f(x,y) = x^2 + y^2$

$z = 0^2 + y^2$

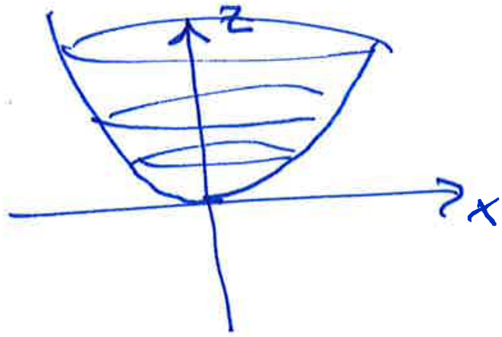
$z = y^2$



y=0: $z = x^2 + y^2$

$z = x^2 + 0^2$

$z = x^2$



Funksjonstyper:

- polynom funksjoner $\quad f(x,y) = x^2 + y^2$
- rasjonale fn. $\quad f(x,y) = \dfrac{xy}{x+y}$
- exp. / log. $\quad f(x,y) = e^{xy}$

$f(x,y) = \ln(x^2 + y^2 + 1)$

- Cobb-Douglas fn:

$f(x,y) = C \cdot x^a y^b \quad (a,b > 0)$

Eks: $f(x,y) = 14 \cdot x^{1.2} y^{0.7}$, Df: $x, y \geq 0$

$z = C \cdot x^a y^b$

$\ln z = \ln(C x^a y^b)$

$= \ln C + \ln(x^a) + \ln(y^b)$

$\boxed{\ln z = \ln C + a \cdot \ln x + b \cdot \ln y}$

(2) Lineære funksjoner

$f(x,y) = ax + by + c$ (polynom av grad ≤ 1)

Eks: $f(x,y) = 3x - y$

Resultat: grafen til $f(x,y)$ er et plan (uten krumning)

$$\Updownarrow$$

$f(x,y)$ er lineær

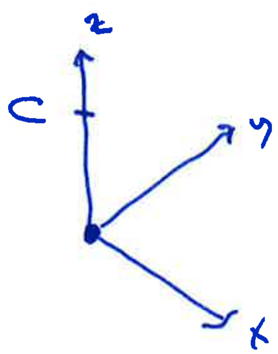
Beskrivelse av grafen til $f(x,y) = ax + by + c$:

i) c = skjæring med z-aksen:

$f(0,0) = c$

og

$(x,y) = (0,0) \leftarrow$ z-aksen



ii) Normalvectoren

$\underline{u}, \underline{v}$: n-vektorer

$\Downarrow$

$\langle \underline{u}, \underline{v} \rangle$: et tall

Defn: Hvis $\underline{u}, \underline{v}$ er to n-vektorer, definerer vi indreproduktet

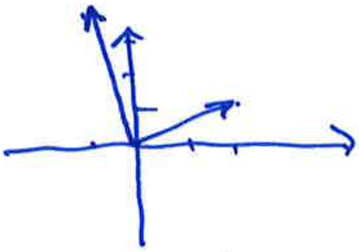
$$\langle \underline{u}, \underline{v} \rangle = \langle \underline{u}, \underline{v} \rangle = \underline{u}^T \cdot \underline{v}$$

matrise-multiplikasjon.

Eks: $\underline{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$$\langle \underline{u}, \underline{v} \rangle = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\rangle = \begin{pmatrix} 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix} = 2 \cdot (-1) + 1 \cdot 3 = \underline{1}$$

$<\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}-1\\3\end{pmatrix}> = 1$



Resultat: $<\underline{u},\underline{v}> = 0$

$\Updownarrow$

$\underline{u} \perp \underline{v}$

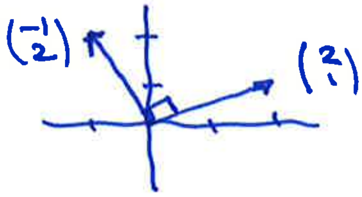
($\underline{u}$ og $\underline{v}$ står normalt eller vinkelrett på hverandre)

$<\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}-1\\2\end{pmatrix}>$

$= (2\ \ 1)\cdot\begin{pmatrix}-1\\2\end{pmatrix} = -2+2=0$

$<\underline{u},\underline{v}> > 0$: vinkel $< 90^\circ$

$<\underline{u},\underline{v}> < 0$: " $> 90^\circ$



Eks: $f(x,y) = 3x-y$

$a=3 \quad b=-1$?

$z = 3x-y$

$0 = 3x-y-z$

$(3\ \ -1\ \ -1)\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$

$<\begin{pmatrix}3\\-1\\-1\end{pmatrix},\begin{pmatrix}x\\y\\z\end{pmatrix}> = 0 \longrightarrow$ Grafen til $z = 3x-y$ er alle vektorer $\begin{pmatrix}x\\y\\z\end{pmatrix}$ slik at $<\begin{pmatrix}3\\-1\\-1\end{pmatrix},\begin{pmatrix}x\\y\\z\end{pmatrix}> = 0$

$\underline{n} = \begin{pmatrix}3\\-1\\-1\end{pmatrix}$ kalles <u>normalvektoren</u> til planet $z = 3x-y$.

$\left.\begin{matrix}a=3\\b=-1\end{matrix}\right\} \longrightarrow \underline{n} = \begin{pmatrix}3\\-1\\-1\end{pmatrix}$

Oppsummering:

Grafen til $f(x,y) = ax + by + c$ er et plan som

- skjærer z-aksen: $z = c$
- har normalvektor $\begin{pmatrix} a \\ b \\ -1 \end{pmatrix}$

③ Partiell derivasjon

$f(x,y)$: funksjon i to variable

Defn:
$$\begin{cases} f'_x = \lim\limits_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h} & \leftarrow y \text{ konstant} \\ f'_y = \lim\limits_{h\to 0} \dfrac{f(x,y+h) - f(x,y)}{h} & \leftarrow x \text{ konstant} \end{cases}$$

Eks: $f(x,y) = x^2 + y^2$

$f'_x = 2x$

$f'_y = 2y$

$f(x,y) = \dfrac{xy}{x+y} = \dfrac{u}{v}$

$$f'_x = \frac{u'_x \cdot v - u \cdot v'_x}{v^2} = \frac{y \cdot (x+y) - xy \cdot (1+0)}{(x+y)^2} = \frac{\cancel{xy} + y^2 - \cancel{xy}}{(x+y)^2} = \frac{y^2}{(x+y)^2}$$

Regneregler:

- de samme som før.
  (men husk at x eller y skal være konstant)

∂ : partiell derivasjon
d : "vanlig" derivasjon

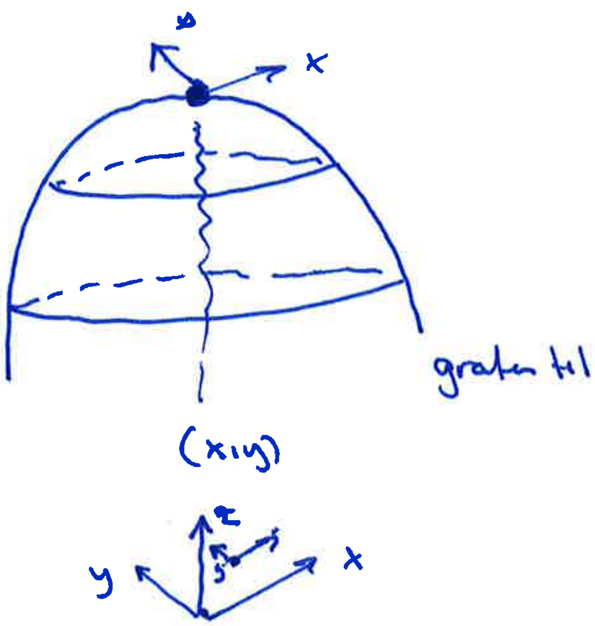
$$f'_y = \frac{u'_y \cdot v - u \cdot v'_y}{v^2} = \frac{x \cdot (x+y) - xy \cdot 1}{(x+y)^2} = \frac{x^2 + \cancel{xy} - \cancel{xy}}{(x+y)^2} = \frac{x^2}{(x+y)^2}$$

Skrivemåte:

$f'_x = f'_1 = \dfrac{\partial f}{\partial x} = (\ldots)'_x$

$f'_y = f'_2 = \dfrac{\partial f}{\partial y} = (\ldots)'_y$

Tolkning:



$(x,y) \rightsquigarrow (x+h,y)$ x-retn

$(x,y+h)$

y-retn.

$\underline{f'_x}$: stigningstall til tangenten når vi beveger oss i x-retn.

$\underline{f'_y}$: stigningstall til tangenten når vi beveger oss i y-retn.