Plan:

1. Regneregler for matriser
2. Eksamen 06/2016, Oppgave 3

Fagoppgave MET11806 } utlevert fredag kl 09. frist neste fredag kl 12.

"Digital innlevering" (Digi Ex)

Ekstra veiledning fredag kl 14–17 i D1-080.

① Regning med matriser

a) Matrisemultiplikasjon:

$$\underset{m\times n}{A} \cdot \underset{n\times p}{B} \longrightarrow \underset{m\times p}{AB}$$

Eks: $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} 17 & 6 \\ -1 & 7 \end{pmatrix}$

Husk:

- $\boxed{AB \neq BA}$
- de fleste andre regneregler er som i "vanlig" algebra

$A \cdot (B+C) = A \cdot B + A \cdot C$

Potenser:

$$\underset{n\times n}{A} \longrightarrow A^m \quad \text{dvs} \quad A^2, A^3, A^4, \ldots$$

Identitetsmatriser:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$A \cdot I = A$

$I \cdot A = A$

"som tallet 1"

b) Transponering:

$$\underset{m\times n}{A} \longrightarrow \underset{n\times m}{A^T}$$

Defn: A kalles symmetrisk hvis $A^T = A$

Eks:

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 7 & -1 \end{pmatrix}$

$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 7 \\ 3 & -1 \end{pmatrix}$

Eks:

$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ symmetrisk

$A^T = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

Regneregler for transponering:

i) $(A^T)^T = A$ ii) $(A \pm B)^T = A^T \pm B^T$

iii) $(c \cdot A)^T = c \cdot A^T$ iv) $\boxed{(AB)^T = B^T \cdot A^T}$

c) Inverse matriser

Defn: Den inverse matrisen til A er en matrise A^{-1} slik at

$$A \cdot A^{-1} = I$$

og

$$A^{-1} \cdot A = I$$

Resultat: $\boxed{A^{-1} \text{ fins (A er invertibel)} \iff |A| \neq 0}$

Utregning av A^{-1}:

$n=2$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

hvis $|A| = ad-bc \neq 0$.

Generell formel: $(n \geq 2)$

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A) = \frac{1}{|A|} \cdot \underbrace{\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T}_{\text{Kofaktor-matrisen}}$$

$C_{ij} = (-1)^{i+j} \cdot M_{ij}$

(C_{ij} er kofaktorer)

Eks: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$

$|A| = -1 \cdot \begin{vmatrix} 1 & 1 \\ 3 & 9 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} - 4 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$

$= -(9-3) + 2(9-1) - 4(3-1)$

$= -6 + 16 - 8 = \underline{\underline{2}} \neq 0$

(A^{-1} fins)

$C_{11} = 6 \quad C_{12} = -5 \quad C_{13} = 1$

$C_{21} = -6 \quad C_{22} = 8 \quad C_{23} = -2$

$C_{31} = 2 \quad C_{32} = -3 \quad C_{33} = 1$

$$A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{2} \cdot \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}^T$$

$$= \underline{\underline{\frac{1}{2} \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}}}$$

Eks:

$x + y + z = 17$
$x + 2y + 4z = 42$
$x + 3y + 9z = 77$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 17 \\ 42 \\ 77 \end{pmatrix}$$

$A \cdot \underline{x} = \underline{b}$

matriseform

$A^{-1} \cdot A\underline{x} = A^{-1} \cdot \underline{b}$

$I\underline{x} = A^{-1}\underline{b}$

$\underline{x} = A^{-1} \cdot \underline{b}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 17 \\ 42 \\ 77 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 6 \cdot 17 - 6 \cdot 42 + 2 \cdot 77 \\ -5 \cdot 17 + 8 \cdot 42 - 3 \cdot 77 \\ 1 \cdot 17 - 2 \cdot 42 + 1 \cdot 77 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 \\ 10 \\ 5 \end{pmatrix}}}$$

Alternativ metode for å regne ut A^{-1}:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

Ser på: $(A \mid I)$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right) \begin{matrix} \\ -1 \\ -1 \end{matrix} \rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{array}\right) \begin{matrix} \\ \\ -2 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right) \cdot \frac{1}{2} \rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right) \begin{matrix} -1 \\ -3 \\ \end{matrix}$$

fortsetter til vi finner
redusert trappeform
(alle pivoter = 1,
null over alle pivoter)

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1/2 & 1 & -1/2 \\ 0 & 1 & 0 & -5/2 & 4 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right) \begin{matrix} \\ -1 \\ \end{matrix} \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -5/2 & 4 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right)$$

$$\uparrow I \qquad \uparrow A^{-1}$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix} \checkmark$$

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Regneregler for A^{-1}:

i) $(A^{-1})^{-1} = A$

ii) $\boxed{(AB)^{-1} = B^{-1} \cdot A^{-1}}$

iii) $(c \cdot A)^{-1} = \frac{1}{c} \cdot A^{-1}$ (c er et tall)

Eks: $(AB)\, B^{-1} A^{-1} = A \cancel{B} \cancel{B^{-1}} A^{-1} = A \cdot I \cdot A^{-1}$

$= A A^{-1} = I$

$AB \cdot A^{-1} B^{-1} \neq I$

Regneregler for determinat:

i) $|A^T| = |A|$ ← fordi vi kan utvikle determinanten langs en rad eller en kolonne

ii) Radoperasjon: $A \to B$

- Bytter to rader: $|B| = -|A|$
- Multiplisrer en rad med $c \neq 0$: $|B| = c \cdot |A|$
- Legge til et multippel av en rad til en annen rad: $|B| = |A|$

iii) $|c \cdot A| = c^n \cdot |A|$ hvis A $n \times n$-matrise, c et tall

iv) Hvis en rad (eller en kolonne) er en lineærkombinasjon av de andre radene (eller de andre kolonnene), så er $|A| = 0$.

Eks: $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \Rightarrow |A| = 0$

$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $|A| = 0$

$A = \begin{pmatrix} 1 & 2 & 7 \\ 3 & -1 & 4 \\ 4 & 1 & 11 \end{pmatrix}$ $|A| = 0$

rad(3) = rad(1) + rad(2)

$A = \begin{pmatrix} 1 & 2 & -1 \\ 7 & 14 & 3 \\ 4 & 8 & 7 \end{pmatrix}$ $|A| = 0$

kolonne(2) = 2 · kolonne(1)

Eks: Når er $\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$?

$\boxed{a = 1 \Rightarrow |\ldots| = 0}$

$\det = a(a^2-1) - 1 \cdot (a-1) + 1 \cdot (1-a) = 0$

$a^3 - a - a + 1 + 1 - a = 0$

$a^3 - 3a + 2 = 0$

$(a-1)(a^2 + a - 2) = 0$

v) $\boxed{|A \cdot B| = |A| \cdot |B|}$

vi) $|A^{-1}| = \frac{1}{|A|}$

$A^{-1} \cdot A = I$

$|A^{-1}| \cdot |A| = |I| = 1$

$|A^{-1}| = \frac{1}{|A|}$

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$|I| = 1$

Oppgaveark 24.

5 c) $A(3B-C) + (A-2B)C + 2B(C+2A)$

$= A \cdot 3 \cdot B - AC + AC - 2BC + 2BC + 2B \cdot 2A$

$= 3AB \cancel{- AC} \cancel{+ AC} - \cancel{2BC} + \cancel{2BC} + 4BA$

$= \underline{\underline{3AB + 4BA}}$

e) $(BAB^{-1})^2 \cdot B^2 = (BA\cancel{B^{-1}})(\cancel{B}A\cancel{B^{-1}}) \cdot \cancel{B} \cdot B$

$= BAAB = \underline{\underline{BA^2B}}$

② Eksamen 06/2016, Opps. 1

$$A = \begin{pmatrix} 2-s & 3 & 3 \\ 3 & 2-s & 3 \\ 3 & 3 & 2-s \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 3 \\ s+4 \\ 1-2s \end{pmatrix}$$

a) $\underline{s=8}$:

$$\left(\begin{array}{ccc|c} -6 & +3 & +3 & 3 \\ +3 & -6 & +3 & 12 \\ +3 & +3 & -6 & -15 \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & 3 & -6 & -15 \\ 3 & -6 & 3 & 12 \\ -6 & 3 & 3 & 3 \end{array}\right) \begin{matrix} -1 \\ 2 \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 3 & 3 & -6 & -15 \\ 0 & -9 & 9 & 27 \\ 0 & 9 & -9 & -27 \end{array}\right) 1 \to \left(\begin{array}{ccc|c} 3 & 3 & -6 & -15 \\ 0 & -9 & 9 & 27 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$3x + 3y - 6z = -15$$
$$-9y + 9z = 27$$

z fri

$$\frac{-9y}{-9} = \frac{27 - 9z}{-9} \qquad y = \underline{-3 + z}$$

$$3x = -15 - 3(-3+z) + 6z$$

$$\frac{3x}{3} = \frac{-6 + 3z}{3} \qquad x = \underline{-2 + z}$$

én fri variabel: z

= en frihetsgrad

Løsning:

$(x, y, z) =$

$\underline{(z-2, z-3, z)}$

med z fri

b) $\begin{vmatrix} 2-s & 3 & 3 \\ 3 & 2-s & 3 \\ 3 & 3 & 2-s \end{vmatrix}$

$$= (2-s)\cdot\left((2-s)^2-9\right) - 3\left(3(2-s)-9\right) + 3\left(9-3(2-s)\right)$$

$$= \underbrace{(2-s)\cdot\left((2-s)^2-9\right)}_{(2-s)(s^2-4s-5)} \underbrace{-3(-3-3s)+3(3+3s)}_{9(1+s)\ +\ 9(1+s)}$$

$$= (2-s)(1+s)(s-5) + 18(1+s)$$

$$= (1+s)\cdot\left[(2-s)(s-5)+18\right]$$

$$= (1+s)\left(-s^2+7s+8\right) = (1+s)(s-8)(s+1)\cdot(-1)$$

$$= -(s+1)^2(s-8)$$

$$-s^2+7s+8=0$$

$$s = \frac{-7 \pm \sqrt{49-4(-1)\cdot 8}}{-2}$$

$$= \frac{-7\pm 9}{-2} = 8, -1$$

c) $\underline{s=0}$: $A = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$

$$A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A) = \frac{1}{8} \begin{pmatrix} -5 & 3 & 3 \\ 3 & -5 & 3 \\ 3 & 3 & -5 \end{pmatrix}^T = \underline{\underline{\frac{1}{8} \begin{pmatrix} -5 & 3 & 3 \\ 3 & -5 & 3 \\ 3 & 3 & -5 \end{pmatrix}}}$$

$\underline{\text{Løsning:}}$ $\begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$

$$A \cdot \underline{x} = \underline{b}$$
$$\cancel{A^{-1} A} \underline{x} = A^{-1} \underline{b}$$
$$\underline{x} = A^{-1} \cdot \underline{b}$$
$$= \frac{1}{8} \begin{pmatrix} -5 & 3 & 3 \\ 3 & -5 & 3 \\ 3 & 3 & -5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 0 \\ -8 \\ 16 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}}}$$

d) $\underline{\text{Systemet har én løsning} \iff |A| \neq 0}$

$|A| = -(s+1)^2 \cdot (s-8)$ fra b)

$|A| = 0 \iff s = -1$ eller $s = 8$

$|A| \neq 0 \iff s \neq -1$ og $s \neq 8$

$\underline{\text{Konklusjon:}}$

$\underline{\underline{\text{eksakt én løsn. for } s \neq -1, 8}}$

$s \neq -1, 8$: Kramers regel

$$|A| = -(s+1)^2(s-8) \qquad A = \begin{pmatrix} 2-s & 3 & 3 \\ 3 & 2-s & 3 \\ 3 & 3 & 2-s \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 3 \\ s+4 \\ 1-2s \end{pmatrix}$$

$$x = \frac{|A_1(\underline{b})|}{|A|} = \frac{0}{-(s+1)^2(s-8)} = \underline{\underline{0}}$$

$$\begin{vmatrix} 3 & 3 & 3 \\ s+4 & 2-s & 3 \\ 1-2s & 3 & 2-s \end{vmatrix} = 3\left((2-s)^2-9\right) - 3\left((s+4)(2-s) - 3(1-2s)\right) + 3\left((s+4)\cdot 3 - (2-s)(1-2s)\right)$$

$$= 3(\cancel{s^2} \cancel{-4s} - 5) - 3(\cancel{-s^2} \cancel{+4s} + 5) + 3(\cancel{-2s^2} + 8s + 10)$$

$$= \cancel{24s} - 15 - 15 + 30 = \underline{0}$$

Skulle kun finne x-koordinaten til løsningen (x, y, z); ikke y og z.