| Solutions | Mock Midterm exam in GRA 6035 Mathematics |
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| Date | Fri October 4th, 2024 at 1000 - 1100 |

**Question 1.**
We use Gaussian elimination:

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 2 \\ 4 & 6 & 10 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -2 & -2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Since $A$ has three pivot positions, we have that $\text{rk}\, A = 3$.

**Question 2.**
We form the matrix with the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ as column vectors, and see that we get the matrix $A$ from Question 1. Hence the Gaussian elimination above shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a base of $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.

**Question 3.**
We use Gaussian elimination to solve the equation $x\mathbf{v}_2 + y\mathbf{v}_3 = \mathbf{v}_1$:

$$\left(\begin{array}{cc|c} 2 & t & 1 \\ t & 8 & 2 \\ 1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & t & 1 \\ t & 8 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & t+4 & -1 \\ 0 & 8+2t & 2-t \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & t+4 & -1 \\ 0 & 0 & 4-t \end{array}\right)$$

We see that for $t = 4$ there is a unique solution, but for $t \neq 4$ there are no solutions since there is a pivot in the last column. Hence $\mathbf{v}_1$ is in $\text{span}(\mathbf{v}_2, \mathbf{v}_3)$ if and only if $t = 4$.

**Question 4.**
The Markov chain is regular since $A > 0$ is a positive matrix (all entries of $A$ are strictly positive), and we compute the eigenvectors in $E_1$:

$$E_1 = \text{Null} \begin{pmatrix} -0.48 & 0.16 \\ 0.48 & -0.16 \end{pmatrix} = \text{Null} \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix}$$

Hence $\mathbf{v}_1 = (1, 3)$ is a base of $E_1$, and the equilibrium state of the Markov chain is therefore the unique state vector $\mathbf{v} = (1/4, 3/4)$ in $E_1$.

**Question 5.**
Since $A$ is upper triangular, the eigenvalues of $A$ are the diagonal entries $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$. Hence $A$ has three eigenvalues (counted with multiplicity), and $A$ is diagonalizable if and only if $\dim E_1 = 2$ since $\lambda = 1$ has multiplicity two, and $\lambda = 2$ has multiplicity one. We compute $\dim E_1$:

$$E_1 = \text{Null} \begin{pmatrix} 0 & s & 1 \\ 0 & 0 & s \\ 0 & 0 & 1 \end{pmatrix} = \text{Null} \begin{pmatrix} 0 & s & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

If $s \neq 0$, the matrix is in echelon form, and $\dim E_1 = 3 - 2 = 1$ since the rank is two and only the first variable is free. If $s = 0$, then $\dim E_1 = 3 - 1 = 2$ since the matrix has rank one and the first two variables are free. Hence $A$ is diagonalizable if and only if $s = 0$.

**Question 6.**
The quadratic form $f(x, y, z) = x^2 + 4xy + 6xz + 3y^2 - 10yz + 8z^2$ has symmetric matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & -5 \\ 3 & -5 & 8 \end{pmatrix}$$

Since $D_1 = 1 > 0$ and $D_2 = 3 - 4 = -1 < 0$, the quadratic form $f$ is indefinite.

**Question 7.**

The leading principal minors of $A$ are $D_1 = 3$, $D_2 = 8$, and $D_3 = -2(8) + 7(8) = 40$. Hence $A$ is positive definite, and $f$ is convex since $H(f) = 2A$ is also positive definite. The stationary points of $f$ are given by $f'(\mathbf{x}) = 2A\mathbf{x} + B^T = \mathbf{0}$. This can be written $A\mathbf{x} = -B^T/2$ where $-B^T/2 = (3, -2, 1)$ as a column vector. We solve the linear system using Gaussian elimination:

$$\left(\begin{array}{ccc|c} 3 & 2 & -2 & 3 \\ 2 & 4 & 0 & -2 \\ -2 & 0 & 7 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & -2 & 5 \\ 2 & 4 & 0 & -2 \\ -2 & 0 & 7 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & -2 & 5 \\ 0 & 8 & 4 & -12 \\ 0 & -4 & 3 & 11 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & -2 & 5 \\ 0 & 8 & 4 & -12 \\ 0 & 0 & 5 & 5 \end{array}\right)$$

Back substitution gives $z = 1$, $y = -2$, and $x = 3$. Since $f$ is convex, it has the minimum point $(x, y, z) = (3, -2, 1)$ and minimum value $f(3, -2, 1) = -14$, and no maximum value. The range of $f$ is $[-14, \infty)$.

**Question 8.**

We see that $\mathbf{v}$ is an eigenvector of $A$ with eigenvalue $\lambda = 5$ since

$$A\mathbf{v} = \begin{pmatrix} 3 & 2 & -2 \\ 2 & 4 & 0 \\ -2 & 0 & 7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 5\mathbf{v}$$

We put $\lambda_1 = 5$ and find the other eigenvalues using that $\mathrm{tr}(A) = 3 + 4 + 7 = 14$ and that $\det(A) = 40$ from Question 7. This gives

$$5 + \lambda_2 + \lambda_3 = 14, \text{ and } 5\lambda_2\lambda_3 = 40 \quad \Rightarrow \quad \lambda_2 + \lambda_3 = 14 - 5 = 9 \text{ and } \lambda_2\lambda_3 = 40/5 = 8$$

We see that $\lambda_2 = 1$ and $\lambda_3 = 8$ since $1 + 8 = 9$ and $1 \cdot 8 = 8$, and conclude that the eigenvalues of $A$ are $\lambda_1 = 5, \lambda_2 = 1, \lambda_3 = 8$.