Solutions	Mock midterm exam in GRA 6035 Mathematics
Date	October 12th, 2023 at 1700 - 1800

Question 1.
We have that $\text{rk}(A) = 2$ since the third row is the sum of the first two rows, and $M_{12,12} = 3-2 = 1 \neq 0$. Alternatively, we may use Gaussian elimination to find the rank:

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 2 & 3 & -1 & 3 \\ 3 & 4 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Question 2.
We can write $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in infinitely many ways, since the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{v}_4$ can be solved using the same elementary row operations as in Question 1, which can be written

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 3 & -1 & 3 \\ 3 & 4 & 0 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and we see from the pivot positions that there is one degree of freedom and infinitely many solutions for (x_1, x_2, x_3).

Question 3.
The equilibrium state is $\mathbf{v} = (1/4, 3/4)$ since the Markov chain is regular, and the eigenvector $(1, 3)$ is a base of the eigenspace

$$E_1 = \text{Null}\begin{pmatrix} -0.21 & 0.07 \\ 0.21 & -0.07 \end{pmatrix} = \text{Null}\begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y/3 \\ y \end{pmatrix} = \frac{y}{3} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

with $1/4 \cdot (1, 3) = (1/4, 3/4)$ as the unique state vector in E_1.

Question 4.
The quadratic form q is positive definite since its symmetric matrix A is given by

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 9 & 13 \\ 6 & 13 & 20 \end{pmatrix}$$

with $D_1 = 2 > 0,\ D_2 = 18 - 16 = 2 > 0,\ D_3 = 6(52 - 54) - 13(26 - 24) + 20(2) = 2 > 0.$

Question 5.
The determinant is $\det(A) = 0$ since the linear system has two distinct solutions, and therefore infinitely many solutions and at least one free variable.

Question 6.
We have that $\mathbf{v}$ is in E_4 and $\dim E_4 = 1$: We have that $\mathbf{v}$ is an eigenvector with eigenvalue $\lambda = 4$ since

$$A \cdot \mathbf{v} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 4 \end{pmatrix} = 4\mathbf{v}$$

and $\dim E_4 = \dim \text{Null}(A - 4I) = 3 - \text{rk}(A - 4I) = 3 - 2 = 1$ since $A - 4I$ has two pivot positions:

$$A - 4I = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Question 7.

The minimum value is $f_{\min} = 5$: In fact, $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C$ with

$$A = \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & -2 & -2 \end{pmatrix}, \quad C = 7$$

where A is positive definite since $D_1 = 1$, $D_2 = 1 - 1/4 = 3/4$, and $D_3 = -1/2(1/2) + 1(3/4) = 1/2$ are positive. This means that f is convex with a single stationary point, given by

$$2A\mathbf{x} + B^T = \mathbf{0} \quad \Rightarrow \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

Since $(x, y, z) = (1, 0, 1)$ is a solution, $f_{\min} = f(1, 0, 1) = 5$.

Question 8.

The matrix A is diagonalizable since it has three distinct eigenvalues: The characteristic equation

$$\begin{vmatrix} -\lambda & 1 & 5 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

gives $-\lambda(\lambda^2 - 1) - 1(-\lambda - 1) + 5(1 + \lambda) = -\lambda(\lambda + 1)(\lambda - 1) + 6(\lambda + 1) = 0$. This gives $\lambda = -1$ or $-\lambda^2 + \lambda + 6$. The eigenvalues of A are $\lambda = -1$, $\lambda = -2$, and $\lambda = 3$.