Solutions	Mock Midterm exam
Date	October 2022

Solutions to Exercise problems

Question 1.
We have $\dim \text{Null}(A) = n - \text{rk}(A) = 4 - 2 = 2$ since A has an echelon form with two pivots:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Question 2.
Yes, the point $(1, 1, -2)$ is a solution of the linear system with $x = y = 1$: Since $(2, 4, 0)$ and $(3, 7, 2)$ are solutions of a linear system, all points on the line through these points are also solutions by the linear property:

$$(x, y, z) = (2, 4, 0) + t(1, 3, 2) = (2 + t, 4 + 3t, 2t) \quad \Rightarrow \quad (x, y, z) = (1, 1, -2) \text{ when } t = -1$$

Question 3.
The equilibrium state is $\mathbf{v} = (1/3, 2/3)$ since the Markov chain is regular, and the eigenvector $(1, 2)$ is a base of the eigenspace

$$E_1 = \text{Null}\begin{pmatrix} -0.26 & 0.13 \\ 0.26 & -0.13 \end{pmatrix} = \text{Null}\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

with $1/3 \cdot (1, 2) = (1/3, 2/3)$ as the unique state vector in E_1.

Question 4.
No, $\mathbf{v}$ is not in $\text{Col}(A)$ since Gaussian elimination gives

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 3 \\ 5 & 4 & 6 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & -4 & 1 \\ 0 & -1 & -4 & -4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & -4 & -1 \\ 0 & 0 & 0 & -5 \end{array}\right)$$

and this linear system has no solutions.

Question 5.
The orthogonal projection is $\text{proj}_{\mathbf{w}}(\mathbf{v}) = (1, 1, 1, 1)$ since

$$\text{proj}_{\mathbf{w}}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w} = \frac{4}{4} \cdot (1, 1, 1, 1) = (1, 1, 1, 1)$$

Question 6.
We have that $\dim \text{Null}(A) = n - \text{rk}(A) = 5 - 3 = 2$ since it follows from the values of the minors that A has three pivot positions at $(1, 1)$, $(2, 2)$, and $(3, 3)$, and no pivot positions in the last row.

Question 7.
The vectors are linearly independent for all values of t, since there is no value of t making these minors zero simultaneously:

$$\begin{vmatrix} t & 3 \\ 2 & 6 \end{vmatrix} = 6t - 6 = 6(t - 1), \qquad \begin{vmatrix} 2 & 6 \\ 3 & t \end{vmatrix} = 2t - 18 = 2(t - 9)$$

Question 8.
The characteristic equation of A is $-\lambda^3 + 14\lambda^2 - 21\lambda = 0$ since $c_1 = \text{tr}(A) = 14$, $c_2 = 7 + 7 + 7 = 21$, and $c_3 = |A| = 0$ (the third row of A is the sum of the other two rows).

Question 9.
The equilibrium state is $\mathbf{v} = (5/20, 7/20, 8/20)$ since the Markov chain is regular, and the eigenvector $(5, 7, 8)$ is a base of the eigenspace

$$E_1 = \text{Null}\begin{pmatrix} -0.6 & 0.2 & 0.2 \\ 0.4 & -0.4 & 0.1 \\ 0.2 & 0.2 & -0.3 \end{pmatrix} = \text{Null}\begin{pmatrix} 2 & 2 & -3 \\ -6 & 2 & 2 \\ 4 & -4 & 1 \end{pmatrix} = \text{Null}\begin{pmatrix} 2 & 2 & -3 \\ 0 & 8 & -7 \\ 0 & -8 & 7 \end{pmatrix}$$

with $1/20 \cdot (5, 7, 8) = (5/20, 7/20, 8/20)$ as the unique state vector in E_1 since $5 + 7 + 8 = 20$.

Question 10.
The matrix A diagonalizable when $s \neq 1$, since the eigenvalues of A is $\lambda = 3, s, 1$, and therefore A is diagonalizable for $s \neq 1, 3$. Moreover, $\dim E_3 = 2 = m$ when $s = 3$ and $\dim E_1 = 1 < m$ when $s = 1$ in the cases of multiplicity $m = 2$.

Question 11.
The rank of A is $\text{rk}\, A = 1$ since $\dim E_0 = \dim \text{Null}(A) = 2$ and $\dim \text{Null}(A) = 3 - \text{rk}\, A$.

Question 12.
The quadratic form q is positive semidefinite by the RRC since its symmetric matrix A has $D_1 = 2$, $D_2 = 7$, and $D_3 = 0$ (the last row of A is the sum of the two other rows), hence $\text{rk}\, A = 2$:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 5 \\ 3 & 5 & 8 \end{pmatrix}$$

Question 13.
The rank of A is $\text{rk}\, A = 3$ since it has an echelon form with three pivots:

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Question 14.
We have $\dim V = \dim \text{Null}(A) = 5 - \text{rk}\, A = 5 - 4 = 1$ since $V = \text{Null}(A)$ of the matrix A of rank 4:

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -4 & -3 \end{pmatrix}$$

Question 15.
The rank of A is $\text{rk}\, A = 1$ if $t = 1$, and $\text{rk}\, A = 3$ if $t \neq 1$, since A has an echelon form

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & t & t^2 & t^3 \\ 1 & 1 & t & t^2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & t-1 & t^2-1 & t^3-1 \\ 0 & 0 & t-1 & t^2-1 \end{pmatrix}$$

with marked pivots if $t \neq 1$, and just the pivot in the first row if $t = 1$.

Question 16.
The eigenvalues of A are $\lambda_1 = \lambda_2 = 3$ and $\lambda_3 = 7$ since the characteristic equation of A is

$$\begin{vmatrix} 5-\lambda & 0 & 2 \\ 0 & 3-\lambda & 0 \\ 2 & 0 & 5-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 10\lambda + 21) = -(\lambda-3)(\lambda-3)(\lambda-7) = 0$$

Question 17.
We have that $\lambda = -1$ is an eigenvalue of A with multiplicty $m = 3$ since $A - \lambda I = A + I$ has rank $\text{rk}(A + I) = 1$:

$$A + I = \begin{pmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore $\dim E_{-1} = 4 - 1 = 3$, and $m = \dim E_{-1} = 3$ since A is symmetric.

Question 18.
The quadratic form q is indefinite since the principal 2-minor $\Delta_2 = M_{14,14} = 0 - 1 = -1$ of the symmetric matrix A is negative:

$$A = \begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 6 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

Question 19.
An orthonormal base of $\text{Null}(A+2I)$ is given by $\{1/\sqrt{2} \cdot (-1, 0, 1), 1/\sqrt{6} \cdot (1, -2, 1)\}$, since the vectors $\mathbf{v}_1 = (-1, 0, 1)$ and $\mathbf{v}_2 = (0, -1, 1)$ is base of the nullspace of $A + 2I$, which has echelon form

$$A + 2I = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\mathbf{v}_2' = \mathbf{v}_2 - \text{proj}_{\mathbf{v}_1}(\mathbf{v}_2) = (0, -1, 1) - 1/2 \cdot (-1, 0, 1) = 1/2 \cdot (1, -2, 1)$ is orthogonal to $\mathbf{v}_1$.

Question 20.
The matrix A is indefinite since $\det(A) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = -6 < 0$ means that one or three of the eigenvalues are negative, and $\text{tr}(A) = 4 > 0$ means that not all three eigenvalues can be negative; therefore A has one negative and two positive eigenvalues.

Question 21.
The stationary point $(1, 1, 1)$ is a local minimum point for f since $(1, 1, 1)$ satisfies the FOC's

$$f_x' = 4x^3 - 4yz = 0, \quad f_y = 4y^3 - 4xz = 0, \quad f_z' = 4z^3 - 4xy = 0$$

and the Hessian at $(1, 1, 1)$ is positive definite with $D_1 = 12$, $D_2 = 128$, and $D_3 = 1024$:

$$H(f)(1, 1, 1) = \begin{pmatrix} 12 & -4 & -4 \\ -4 & 12 & -4 \\ -4 & -4 & 12 \end{pmatrix}$$

Question 22.
We have that f is convex if $a \geq 0$ and concave if $a \leq 0$ since $H(f) = -a \cdot H(g)$ and $H(g)$ is negative semidefinite. This means that $H(f)$ is positive semidefinite when $a \geq 0$, and negative semidefinite when $a \leq 0$.