Solutions	Mock Midterm exam in GRA 6035 Mathematics
Date	October 9th, 2020 at 1300 - 1400

Correct answers: A-B-D-C D-A-C-D

Question 1.
We find the pivot positions in A, given by the Gaussian process

$$\left(\begin{array}{cccc|c} 0 & 0 & 3 & 1 & 4 \\ 0 & 0 & 1 & 5 & 7 \\ 1 & 1 & 0 & 5 & 13 \\ 4 & 0 & 0 & 1 & 5 \end{array}\right) \quad \to \quad \left(\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 13 \\ 0 & -4 & 0 & -19 & -47 \\ 0 & 0 & 1 & 5 & 7 \\ 0 & 0 & 0 & -14 & -17 \end{array}\right)$$

Hence there is a unique solution. The correct answer is alternative **A**.

Question 2.
We compute the determinant of A to find out when $\operatorname{rk} A$ is maximal:

$$\begin{vmatrix} 1 & t & -t \\ 5 & -t & t \\ 4 & 2 & 0 \end{vmatrix} = 4(t^2 - t^2) - 2(t + 5t) = -12t$$

Hence $\operatorname{rk} A = 3$ when $t \neq 0$. When $t = 0$, the 2-minor $M_{23,12} = 10 + 4t = 10 \neq 0$, and $\operatorname{rk} A = 2$. The correct answer is alternative **B**.

Question 3.
We find the pivot positions of A using the Gaussian process

$$\begin{pmatrix} 3 & 1 & 7 & 0 \\ 2 & 0 & 5 & 8 \\ 3 & 5 & 5 & 2 \end{pmatrix} \quad \to \quad \begin{pmatrix} 3 & 1 & 7 & 0 \\ 0 & -2 & 1 & 24 \\ 0 & 0 & 0 & 50 \end{pmatrix}$$

Hence $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a base of V, and $\mathbf{v}_3$ is a linear combination of the vectors in this base. The correct answer is alternative **D**.

Question 4.
The symmetric matrix of the quadratic form $f(x, y, z) = x^2 + 4xy - 2xz + 5y^2 - 4yz + z^2$ is given by

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 5 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

The leading principal minors are $D_1 = 1$, $D_2 = 5 - 4 = 1$, $D_3 = -1(1) + 2(0) + 1(1) = 0$. Since A has $\operatorname{rk} A = 2$, it follows by the reduced rank criterion that A and f are positive semidefinite, but not positive definite. The correct answer is alternative **C**.

Question 5.
We compute the eigenvalues of A by solving the characteristic equations $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} -\lambda & 0 & 2 \\ 4 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

We compute the determinant by cofactor expansion along the third column, which gives

$$2(4) - \lambda(\lambda^2) = 0 \quad \Rightarrow \quad \lambda^3 = 8$$

Hence $\lambda = \sqrt[3]{8} = 2$ is the unique real eigenvalue. In fact, we have $\lambda^3 - 8 = (\lambda - 2)(\lambda^2 + 2\lambda + 4)$, and the quadratic factor has no (real) zeros. The correct answer is alternative **D**.

Question 6.

The function $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$ has first order partial derivatives and first order conditions given by

$$f'_x = 6x^2 + y^2 + 10x = 0, \quad f'_y = 2xy + 2y = 0$$

From the second equation we get $2y(x+1) = 0$, so $y = 0$ or $x = -1$. In the first case, we get $6x^2 + 10x = 0$ from the first equation, and $x = 0$ or $x = -5/3$. In the second case, we get $y^2 - 4 = 0$, or $y = \pm 2$. There are therefore four stationary points $(0,0)$, $(-5/3, 0)$, $(-1, \pm 2)$. The Hessian matrix at the first two points are given by

$$H(f) = \begin{pmatrix} 12x+10 & 2y \\ 2y & 2x+2 \end{pmatrix} \quad \Rightarrow \quad H(f)(0,0) = \begin{pmatrix} 10 & 0 \\ 0 & 2 \end{pmatrix}, \quad H(f)(-5/3,0) = \begin{pmatrix} -10 & 0 \\ 0 & -4/3 \end{pmatrix}$$

This means that $(0,0)$ is a local min and $(-5/3, 0)$ is a local max. We could also check the last two stationary points, which are saddle points, but it is not necessary to answer the question. The correct answer is alternative **A**.

Question 7.

The Hessian matrix of the function $f(x,y,z) = x^2 + 4xy - 2xz + 5y^2 - 4yz + hz^2 + z^4$ is given by

$$H(f) = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 10 & -4 \\ -2 & -4 & 2h+12z^2 \end{pmatrix}$$

The leading principal minors are $D_1 = 2$, $D_2 = 20 - 16 = 4$, $D_3 = -2(4) + 4(0) + (2h + 12z^2)(4)$, which gives $D_3 = 48z^2 + 8h - 8$. When $h \geq 1$, $H(f)$ is positive semidefinite for all (x,y,z) since $D_1, D_2 > 0$ and $D_3 \geq 0$ (we use the RRC in case $D_3 = 0$). When $h < 1$, $D_3 < 0$ when $z = 0$ so the matrix is indefinite. This means that f is convex for $h \geq 1$ but not for $h < 1$. The correct answer is alternative **C**.

Question 8.

We compute the eigenvalues of A by solving the characteristic equations $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} -\lambda & 0 & 2 \\ 4 & -\lambda & 0 \\ 0 & s^3 & -\lambda \end{vmatrix} = 0$$

We compute the determinant by cofactor expansion along the third column, which gives

$$2(4s^3) - \lambda(\lambda^2) = 0 \quad \Rightarrow \quad 8s^3 - \lambda^3 = 0$$

Hence $\lambda = \sqrt[3]{8s^3} = 2s$ is one solution, and polynomial division gives

$$\lambda^3 - 8s^3 = (\lambda - 2s)(\lambda^2 + 2s\lambda + 4s^2)$$

Since the quadratic factor can be written $(\lambda + s)^2 + 3s^2$, it has no (real) zeros when $s \neq 0$. Hence, there is only one eigenvalue $\lambda = 2s$ of multiplicity one in case $s \neq 0$, and A is not diagonalizable. In the case $s = 0$, we get $\lambda^3 = 0$, so $\lambda = 0$ is an eigenvalue of multiplicity 3. But $\operatorname{rk} A = 2$ in this case, so A is not diagonalizable for $s = 0$ either. The correct answer is alternative **D**.