Solutions: Mock Exam 22/11/2010

1. $A = \begin{pmatrix} 1 & 1 & -4 \\ 0 & t+2 & t-8 \\ 0 & -5 & 5 \end{pmatrix}$

a) Compute det(A) and rk(A):

$$|A| = 1 \cdot \left((t+2) \cdot 5 - (t-8)(-5) \right) = 5t+10+5t-40 = \underline{\underline{10t-30}}$$

$$\text{rk } A = \begin{cases} 3, & t \neq 3 \quad \text{since } |A|=0 \iff t=3 \\ 2, & t=3 \quad \text{since } \begin{vmatrix} 1 & 1 \\ 0 & t+2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 5 \end{vmatrix} = 5 \neq 0 \end{cases}$$

b) Eigenvalues of A?

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & -4 \\ 0 & t+2-\lambda & t-8 \\ 0 & -5 & 5-\lambda \end{vmatrix} = (1-\lambda)\left((t+2-\lambda)(5-\lambda) + 5(t-8) \right)$$

$$= (1-\lambda) \cdot \left(\lambda^2 - (t+7)\lambda + (5t+10+5t-40) \right)$$

$$= (1-\lambda)\left(\lambda^2 - (t+7)\lambda + (10t-30)\right) = 0$$

$$\lambda = 1 \quad \text{or} \quad \lambda = \frac{(t+7) \pm \sqrt{(t+7)^2 - 4 \cdot (10t-30)}}{2} = \frac{(t+7) \pm \sqrt{t^2 - 26t + 169}}{2}$$

$$= \frac{(t+7) \pm (t-13)}{2} = t-3, \ 10$$

$$\underline{\underline{\lambda = 1, \ \lambda = t-3, \ \lambda = 10}}$$

c) When is A diagonizable:

$t=4$, $t=13$: Double roots (two eigenvalues)

$t \neq 4, 13$: Three distinct eigenvalues

A is diagonalizable for $t \neq 4, 13$ since there are three distinct eigenvalues

For $t=4$: $\lambda = 1$ (double root) $\lambda = 10$

Check the double root $\lambda = 1$:

$$A - \lambda I = \begin{pmatrix} 0 & 1 & -4 \\ 0 & t+2-1 & t-8 \\ 0 & -5 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -4 \\ 0 & 5 & -4 \\ 0 & -5 & 4 \end{pmatrix}$$

$rk = 2$ since $\begin{vmatrix} 1 & -4 \\ 5 & -4 \end{vmatrix} \neq 0$

free var. $= 1$

Since $\lambda = 1$ is a double root, this means that A is not diagonalizable for $t=4$.

For $t=13$: $\lambda = 1$, $\lambda = 10$ (double root)

Check the double root $\lambda = 10$:

$$A - \lambda I = \begin{pmatrix} -12 & 1 & -4 \\ 0 & t+2-10 & t-8 \\ 0 & -5 & -5 \end{pmatrix} = \begin{pmatrix} -12 & 1 & -4 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{pmatrix}$$

$rk = 2$ since $\begin{vmatrix} -12 & 1 \\ 0 & 5 \end{vmatrix} \neq 0$

free var. $= 1$

Since $\lambda = 10$ is a double root, this means that A is not diagonalizable for $t = 13$.

Conclusion: A diagonalizable for $t \neq 4, 13$

2.

a) $f(x,y,z) = e^{xy+yz-xz} = e^u$, $u = xy+yz-xz$

$f'_x = 0 \qquad e^u \cdot (y-z) = 0 \qquad y-z=0$

$f'_y = 0 \qquad e^u \cdot (x+z) = 0 \qquad x+z=0$

$f'_z = 0 \qquad e^u (y-x) = 0 \qquad y-x=0$

Stationary pts:

$$\left.\begin{array}{lll} y-z=0 & y=z & y=0 \\ x+z=0 & x=-z & x=0 \\ y-x=0 & y-x=z+z=0 \Rightarrow z=0 & \end{array}\right\} \quad (x,y,z) = \underline{\underline{(0,0,0)}}$$

b) $g(x,y,z) = e^{ax+by+cz} = e^u$, $u = ax+by+cz$

$g'_x = e^u \cdot a \qquad g''_{xx} = e^u \cdot a^2 \qquad g''_{yy} = e^u \cdot b^2 \qquad g''_{zz} = e^u \cdot c^2$

$g'_y = e^u \cdot b \qquad g''_{xy} = e^u \cdot ab \qquad g''_{yz} = e^u \cdot bc$

$g'_z = e^u \cdot c \qquad g''_{xz} = e^u \cdot ac$

$$g'' = \begin{pmatrix} a^2e^u & abe^u & ace^u \\ abe^u & b^2e^u & bce^u \\ ace^u & bce^u & c^2e^u \end{pmatrix} = e^u \cdot \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}$$

When is g convex / concave?

Principal minors:

$D_1 = a^2e^u \qquad (\Delta_1 = a^2e^u, b^2e^u, c^2e^u)$

$D_2 = 0 \qquad (\Delta_2 = 0, 0, 0)$

$D_3 = 0 \qquad (\Delta_3 = 0)$

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Hence $\Delta_i \geq 0$ for all principal minors Δ_i, so g is convex for all a, b, c.

Moreover $\Delta_i \leq 0$ for all principal minors if and only if $a=b=c=0$, so g is concave for $a=b=c=0$

3.

a) $y(1-y) = y'$, $y(0) = 1/2$

$$\frac{1}{y(1-y)} y' = 1 \qquad \text{(separable)}$$

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$$

$$1 = A(1-y) + By$$

$y=0$: $A=1$

$y=1$: $B=1$

$$\int \frac{1}{y(1-y)}\, dy = \int 1\, dt$$

$$\int \frac{1}{y} + \frac{1}{1-y}\, dy = \int 1\, dt$$

$$\ln|y| - \ln|1-y| = t + C_1$$

$$\ln\left|\frac{y}{1-y}\right| = t + C_1$$

$$\frac{y}{1-y} = \pm e^{t+C_1} = C_2 \cdot e^t$$

$$C_2 = \pm e^{C_1}$$

$$y = (1-y)C_2 e^t = C_2 e^t - y C_2 e^t$$

$$y(1 + C_2 e^t) = C_2 e^t \Rightarrow y = \frac{C_2 e^t}{1 + C_2 e^t}$$

$$y(0) = \tfrac{1}{2}: \quad \frac{1}{2} = \frac{C_2}{1 + C_2} \Rightarrow C_2 = 1 \Rightarrow y = \frac{e^t}{1 + e^t}$$

b) $$\left(\ln(t^2+1)-2\right)y' = 2t - \frac{2ty}{1+t^2}$$

$$\left(\frac{2ty}{1+t^2} - 2t\right) + \left(\ln(t^2+1)-2\right)y' = 0$$

$$\boxed{u(y,t) = C}\ \text{, where } \begin{cases} \dfrac{\partial u}{\partial t} = M \\[2mm] \dfrac{\partial u}{\partial y} = N \end{cases}$$

$M + N\cdot y' = 0$

exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

$$\left.\begin{array}{l} \dfrac{\partial M}{\partial y} = \dfrac{2t}{1+t^2} \\[2mm] \dfrac{\partial N}{\partial t} = \dfrac{2t}{t^2+1} \end{array}\right\} \text{ok.}$$

$$\frac{\partial u}{\partial t} = M \Rightarrow$$

$$u = \int M\,dt = \int\left(\frac{2ty}{1+t^2} - 2t\right)dt$$

$$= y\cdot\ln(1+t^2) - t^2 + \varphi(y) \qquad \left(\text{since } \frac{\partial \varphi(y)}{\partial t} = 0\right)$$

$$\frac{\partial u}{\partial y} = \ln(1+t^2) + \varphi'(y) = N = \ln(t^2+1) - 2$$

$$\varphi'(y) = -2$$

$$\varphi(y) = -2y$$

$$u = y\ln(1+t^2) - t^2 - 2y = C$$

$$y\left(\ln(1+t^2) - 2\right) = t^2 + C$$

$$y = \frac{t^2+C}{\ln(t^2+1)-2}$$

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c) $p_{t+2} = \frac{2}{3} p_{t+1} + \frac{1}{3} p_t$, $p_0 = 100$, $p_1 = 102$

$$p_{t+2} - \frac{2}{3} p_{t+1} - \frac{1}{3} p_t = 0$$

$$r^2 - \frac{2}{3} r - \frac{1}{3} = 0$$

$$r = \frac{2/3 \pm \sqrt{4/9 - 4 \cdot (-1/3)}}{2} = \frac{2/3 \pm \sqrt{16/9}}{2} = \frac{2 \pm 4}{6}$$

$$r = 1, -1/3 \Rightarrow p_t = C_1 \cdot 1 + C_2 \cdot (-1/3)^t$$

$p_0 = 100$: $\quad 100 = C_1 + C_2$

$p_1 = 102$: $\quad 102 = C_1 - \frac{1}{3} C_2$

$$-2 = \frac{4}{3} C_2 \Rightarrow C_2 = -\frac{6}{4} = -1.5$$

$$C_1 = 100 - C_2 = 101.5$$

$$p_t = \underline{\underline{101.5 - 1.5 \cdot (-1/3)^t}}$$

4. Max $f(x,y,z) = xy + yz - xz$ subject to $x^2 + y^2 + z^2 \leq 1$.

a) $g(x,y,z) = x^2 + y^2 + z^2 - 1$: $\boxed{g(x,y,z) \leq 0}$

$$\mathcal{L} = f(x,y,z) - \lambda g(x,y,z) = \underline{xy + yz - xz - \lambda(x^2 + y^2 + z^2 - 1)}$$

$$\left.\begin{aligned} \mathcal{L}'_x &= (y - z) - \lambda \cdot 2x = 0 \\ \mathcal{L}'_y &= (x + z) - \lambda \cdot 2y = 0 \\ \mathcal{L}'_z &= (y - x) - \lambda \cdot 2z = 0 \end{aligned}\right\} \text{First order conditions}$$

Note: linear system in x, y, z.

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$$\begin{pmatrix} -2\lambda & 1 & -1 \\ 1 & -2\lambda & 1 \\ -1 & 1 & -2\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solve this linear system for x, y, z:

$$\begin{vmatrix} -2\lambda & 1 & -1 \\ 1 & -2\lambda & 1 \\ -1 & 1 & -2\lambda \end{vmatrix} = -2\lambda(4\lambda^2-1) - 1\cdot(-2\lambda+1) + 1(1-2\lambda)$$

$$= -2\lambda(4\lambda^2-1) + (2\lambda-1) + (2\lambda-1) =$$

$$= -2\lambda(2\lambda-1)(2\lambda+1) + 2\cdot(2\lambda-1) = (2\lambda-1)\cdot\left(-2\lambda(2\lambda+1)+2\right)$$

$$= (2\lambda-1)(-4\lambda^2-2\lambda+2) = 0$$

$$\lambda = 1/2 \quad \text{or} \quad \lambda = \frac{2 \pm \sqrt{4-4\cdot(-4)\cdot 2}}{2\cdot(-4)} = \frac{2\pm 6}{-8} = -1,\ 1/2$$

$\lambda \neq \frac{1}{2}, -1$: $(x,y,z) = (0,0,0)$

$\lambda = \frac{1}{2}$:

$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow[\text{Gauss}]{} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

y, z : free
$x = y - z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y-z \\ y \\ z \end{pmatrix}$$

$\lambda = -1$:

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

z free
$x = z$
$y = -z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -z \\ z \end{pmatrix}$$

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Conclusion:

$\lambda \neq 1/2, -1$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 1/2$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y-z \\ y \\ z \end{pmatrix}$, y, z free

$\lambda = -1$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -z \\ z \end{pmatrix}$, z free

b) Since $\{(x,y,z) : x^2+y^2+z^2 \leq 1\}$ is closed and bounded (a ball of radius 1) and f is cont., the ~~probl~~ problem has a solution by the extreme value theorem.

Kuhn-Tucker conditions:

i) $x^2+y^2+z^2 = 1$, $\lambda \geq 0$ + first order conditions:

By a), $\lambda \neq 1/2, -1$ is not possible (since $x=y=z=0$ is not possible)

$\lambda = -1$ is not possible

$\Rightarrow \lambda = \frac{1}{2}$, $(x,y,z) = (y-z, y, z)$ and $x^2+y^2+z^2=1$.

this gives $(y-z)^2 + y^2 + z^2 = 1$

$y^2 - 2yz + z^2 + y^2 + z^2 = 1$

$2y^2 - 2yz + 2z^2 = 1$

$y^2 - yz + z^2 = 1/2$

All points $(x,y,z;1/2)$ s.t. $x = y-z$ and $y^2 - yz + z^2 = 1/2$ are solutions of Kuhn-Tucker.

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ii) $x^2+y^2+z^2 < 1,\ \lambda = 0$;

$\lambda = 0$ give $(x,y,z) = (0,0,0)$ by a)

$\Rightarrow$ One solution $(0,0,0;0)$

NDCQ:

Case i): NDCQ is $\mathrm{rk}\begin{pmatrix} 2x & 2y & 2z \end{pmatrix} = 1$

This is ok unless $(x,y,z) = (0,0,0)$, and $(x,y,z) \neq (0,0,0)$ in case i).

Case ii): No condition

$\Downarrow$

NDCQ always satisfied.

Conclusion:

There is a solution.

A solution must be either

$(x,y,z;\ 1/2)$ with $x = y - z$, $y^2 - yz + z^2 = 1/2$	$f(x,y,z) = xy + yz - xz$ $= (y-z)y + yz - (y-z)z$ $= y^2 \cancel{-yz} \cancel{+yz} - yz + z^2 = 1/2$
$(0,0,0;0)$	$f(0,0,0) = 0$

Result: $\max\limits_{g(x,y,z)\le 0} f(x,y,z) = \dfrac{1}{2}$