Solutions	Midterm exam in GRA 6035 Mathematics
Date	January 9th, 2023 at 1700 - 1800

Question 1.
We have $\dim \operatorname{Col}(A) = \operatorname{rk}(A) = 3$ since A has an echelon form with three pivots:

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Question 2.
We use the echelon form in Question 1 to find a base of $\operatorname{Null}(A)$: We find that $2w = 0$, or $w = 0$, that z is free, that $y + z = 0$, or $y = -z$, and that $x + y + 2z + w = 0$, or $x = -y - 2z - w = z - 2z = -z$. Hence $(x, y, z, w) = (-z, -z, z, 0) = z(-1, -1, 1, 0)$ is in $\operatorname{Null}(A)$, and $(-1, -1, 1, 0)$ is a base. Hence $-\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = 0$, and this gives that $\mathbf{v}_1 = -\mathbf{v}_2 + \mathbf{v}_3$.

Question 3.
The rank of A is maximal for $s \neq 1, s \neq 2$ since $\operatorname{rk}(A) < 3$ if and only if

$$|A| = \begin{vmatrix} 3 & s & 1 \\ 1 & 1 & s \\ 4 & 3 & 3 \end{vmatrix} = 4(s^2 - 1) - 3(3s - 1) + 3(3 - s) = 4s^2 - 12s + 8 = 4(s - 1)(s - 2) = 0$$

Question 4.
The equilibrium state is $\mathbf{v} = (1/8, 7/8)$ since the Markov chain is regular, and the eigenvector $(1, 7)$ is a base of the eigenspace

$$E_1 = \operatorname{Null} \begin{pmatrix} -0.42 & 0.06 \\ 0.42 & -0.06 \end{pmatrix} = \operatorname{Null} \begin{pmatrix} -7 & 1 \\ 0 & 0 \end{pmatrix}$$

with $1/8 \cdot (1, 7) = (1/8, 7/8)$ as the unique state vector in E_1.

Question 5.
The quadratic form q is indefinite since its symmetric matrix A has $D_2 = 3 - 4 = -1 < 0$:

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 2 & -3 & -2 \\ 1 & -2 & -1 \end{pmatrix}$$

Question 6.
We have that $\mathbf{v}$ is in E_3 since $A \cdot \mathbf{v} = 3\mathbf{v}$, hence $\lambda = 3$.

Question 7.
The Hessian matrix $H(f)$ has leading principal minors $D_1 = 12x^2 + 4 > 0$, $D_2 = 6(12x^2 + 4) > 0$ and $D_3 = 6(12(12x^2 + 4) - (-6)^2) = 6(144x^2 + 12) > 0$ since $H(f)$ is given by

$$H(f) = \begin{pmatrix} 12x^2 + 4 & 0 & -6 \\ 0 & 6 & 0 \\ -6 & 0 & 12 \end{pmatrix}$$

The Hessian $H(f)$ is therefore positive definite for all (x, y, z), hence f is convex (but not concave).

Question 8.
A vector $\mathbf{v} = (x, y, z)$ is orthogonal to $(1, 3, 2)$ if and only if $(x, y, z) \cdot (1, 3, 2) = 0$, or $x + 3y + 2z = 0$. We find two linearly independent solutions by taking y, z as free variables and solving for $x = -3y - 2z$. This gives $(x, y, z) = (-3y - 2z, y, z) = y(-3, 1, 0) + z(-2, 0, 1)$. We may choose $\mathbf{w}_1 = (-3, 1, 0)$ and $\mathbf{w}_2 = (-2, 0, 1)$.