Solutions	Midterm exam in GRA 6035 Mathematics
Date	April 28th, 2022 at 1700 - 1800

Correct answers: D-C-A-D B-D-D-C

Question 1.
We find the pivot positions in A using the Gaussian process

$$\left(\begin{array}{cccc|c} 1 & 0 & 7 & 1 & 13 \\ 3 & 2 & 0 & 2 & 3 \\ 7 & 4 & 7 & 5 & 19 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 7 & 1 & 13 \\ 0 & 2 & -21 & -1 & -36 \\ 0 & 4 & -42 & -2 & -72 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 7 & 1 & 13 \\ 0 & 2 & -21 & -1 & -36 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

From the pivot positions, we see that there are two degrees of freedom. The correct answer is alternative **D**.

Question 2.
We find the pivot positions in A using the Gaussian process

$$\begin{pmatrix} 1 & 2 & 5 \\ 1 & 1 & 4 \\ 2 & 0 & 6 \\ 1 & 3 & a \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 5 \\ 0 & -1 & -1 \\ 0 & -4 & -4 \\ 0 & 1 & a-5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 5 \\ 0 & -1 & -1 \\ 0 & 0 & a-6 \\ 0 & 0 & 0 \end{pmatrix}$$

This shows that the column vectors of A are linearly dependent for $a = 6$, and linearly independent for all other values of a. The correct answer is alternative **C**.

Question 3.
We compute the eigenvalues of A by solving the characteristic equations $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix} = 0$$

We compute the determinant by cofactor expansion along the second row, which gives

$$(2-\lambda) \cdot \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 2\lambda - 8)$$

Since $\lambda^2 - 2\lambda - 8 = 0$ has roots $\lambda = -2$ and $\lambda = 4$, A has three eigenvalues of multiplicity one. The correct answer is alternative **A**.

Question 4.
The symmetric matrix of the quadratic form f is given by

$$A = \begin{pmatrix} 3 & 4 & 1 \\ 4 & 6 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

Since $D_1 = 3$, $D_2 = 18 - 16 = 2$, and $D_3 = 1(8-6) - 2(6-4) = -2$, A is indefinite. The correct answer is alternative **D**.

Question 5.
The Markov chains is regular since its graphs has a path from any state to any other state of length 2; that is, A^2 is a positive matrix. The eigenvectors for $\lambda = 1$ are given by the linear system $(A - I)\mathbf{x} = \mathbf{0}$, and an echelon form of the coefficient matris is given by

$$A - I = \begin{pmatrix} -0.5 & 0.3 & 0 \\ 0 & -0.3 & 0.5 \\ 0.5 & 0 & -0.5 \end{pmatrix} \to \begin{pmatrix} -0.5 & 0.3 & 0 \\ 0 & -0.3 & 0.5 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, we see that $(3, 5, 3)$ is an eigenvector in E_1, and all others are multiples of this vector. Multiplication by $1/11$ gives the state vector $(x, y, z) = (3/11, 5/11, 3/11)$. The correct answer is alternative **B**.

Question 6.
Since f is a quadratic form, $\mathbf{x} = (0, 0, 0)$ is a stationary point. The symmetric matrix of f is given by

$$A = \begin{pmatrix} 3 & 4 & 1 \\ 4 & 6 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Since $D_1 = 3$, $D_2 = 18 - 16 = 2$, $D_3 = 1(8 - 6) - 2(6 - 4) + 1(2) = 0$, A is positive semidefinite but not positive definite by the RRC. It follows that f is convex, and therefore has a global minimum point $\mathbf{x} = (0, 0, 0)$. The correct answer is alternative **D**.

Question 7.
We compute an echelon form of A using elementary row operations, and get

$$A = \begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & a & -7 & -2 \\ 3 & 10 & -7 & 16 & 7 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & -3 & a-15 & -7 & -5 \\ 0 & 4 & 8 & 16 & 10 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & a-9 & 5 & 10 \\ 0 & 0 & 0 & 0 & -10 \end{pmatrix}$$

Hence A has rank 4, with the pivot in the third row in the third column if $a \neq 9$, and in the third column if $a = 9$. The correct answer is alternative **D**.

Question 8.
The characteristic equation can be written $\lambda^2(\lambda + \sqrt{3})(\lambda - \sqrt{3}) = 0$, and $\lambda = 0$ has multiplicity 2. Since $1 \leq \dim E_0 \leq 2$, and A is diagonalizable if $\dim E_0 = 2$, we must have $\dim E_0 = 1$. This means that $\operatorname{rk} A = 4 - \dim E_0 = 3$. The correct answer is alternative **C**.