Solutions	Midterm exam in GRA 6035 Mathematics
Date	January 27th, 2022 at 1700 - 1800

Correct answers: A-D-B-C B-A-D-C

Question 1.
We find the pivot positions in A using the Gaussian process

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 2 & -1 & 3 & 0 & 1 \\ 3 & 0 & 4 & 1 & 4 \end{array}\right) \quad \to \quad \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & -3 & 1 & -2 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Since there is a pivot in the last column, there are no solutions. The correct answer is alternative **A**.

Question 2.
Since $\operatorname{rk} A \leq 3$ for any value of a, the linear system $A\mathbf{x} = \mathbf{0}$ has at least one free variable. This shows that the column vectors of A are linearly dependent for all values of a. The correct answer is alternative **D**.

Question 3.
We compute the eigenvalues of A by solving the characteristic equations $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

We compute the determinant by cofactor expansion along the third row, which gives

$$(3-\lambda) \cdot \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 4\lambda + 3)$$

Since $\lambda^2 - 4\lambda + 3 = 0$ has roots $\lambda = 1$ and $\lambda = 3$, the eigenvalue $\lambda = 3$ has multiplicity two, and $\lambda = 1$ has multiplicity one. The correct answer is alternative **B**.

Question 4.
Eigenvectors for $\lambda = 1$ are given by the linear system $(A - I)\mathbf{x} = \mathbf{0}$, where

$$A - I = \begin{pmatrix} -0.44 & 0.22 \\ 0.44 & -0.22 \end{pmatrix}$$

Therefore, we see that $(1, 2)$ is an eigenvector in E_1, and all others are multiple of this vector. Multiplication by $1/3$ gives the state vector $(x, y) = (1/3, 2/3)$. The correct answer is alternative **C**.

Question 5.
We compute the eigenvalues of A by solving the characteristic equations $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & s-\lambda & 2 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

Cofactor expansion along the middle column gives $(s - \lambda)(\lambda^2 - 4\lambda + 3) = 0$, and the eigenvalues of A are $\lambda = s, 1, 3$. Hence A has three distinct eigenvalues when $s \neq 1, 3$. In case $s = 1$ or $s = 3$, the eigenvalue $\lambda = s$ has multiplicity two, and the eigenspace is the null space of the matrix

$$s = 1: \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \quad s = 3: \quad \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & -1 \end{pmatrix}$$

In each case, there is one free variable y, while $\lambda = s$ has multiplicity two. Hence A is not diagonalizable for $s = 1$ or $s = 3$. The correct answer is alternative **B**.

Question 6.
The symmetric matrix of the quadratic form f is given by

$$A = \begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 7 & 0 \\ -1 & 1 & 0 & 4 \end{pmatrix}$$

Since $D_1 = 3$, $D_2 = 3 - 1 = 2$, $D_3 = 3(7-4) - 1(7-8) + 4(2-4) = 2$, and cofactor expansion along the last row gives

$$D_4 = -(-1) \cdot \begin{vmatrix} 1 & 4 & -1 \\ 1 & 2 & 1 \\ 2 & 7 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \\ 4 & 7 & 0 \end{vmatrix} + 4 \cdot D_3 = -2 - 4 + 8 = 2$$

it follows that A is positive definite. The correct answer is alternative **A**.

Question 7.
Since f is a quadratic form, $\mathbf{x} = (0,0,0)$ is a stationary point. The symmetric matrix of f is given by

$$A = \begin{pmatrix} 3 & 1 & 4 \\ 1 & 1 & 2 \\ 4 & 2 & 7 \end{pmatrix}$$

Since $D_1 = 3$, $D_2 = 3 - 1 = 2$, $D_3 = 3(7-4) - 1(7-8) + 4(2-4) = 2$, A is positive definite. It follows that f is convex, and therefore has a global minimum point $\mathbf{x} = (0,0,0)$. The correct answer is alternative **D**.

Question 8.
Since $|A| = \lambda_1\lambda_2\lambda_3 = -6 < 0$, we either have three negative eigenvalues, or one negative and two positive eigenvalues. Since $\text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 = 4 > 0$, we cannot have three negative eigenvalues, and this means that A is indefinite since it has both positive and negative eigenvalues. The correct answer is alternative **C**.