EVALUATION GUIDELINES - Multiple choice

GRA 60352 Mathematics

Department of Economics

Start date:	08.01.2020	Time 09:00
Finish date:	08.01.2020	Time 10:00

For more information about formalities, see examination paper.

Solutions	Midterm exam in GRA 6035 Mathematics
Date	January 8th, 2020 at 0900 - 1000

Correct answers: C-C-A-D D-B-B-C

Question 1.
We find the pivot positions in A, given by the Gaussian process

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 4 & 4 \\ 0 & 0 & 3 & 7 & 1 \\ 0 & 0 & 6 & 13 & 2 \end{array}\right) \quad \to \quad \left(\begin{array}{cccc|c} 1 & -1 & 1 & 4 & 4 \\ 0 & 0 & 3 & 7 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right)$$

Hence there is one degree of freedom. The correct answer is alternative **C**.

Question 2.
We find the pivot positions in A, given by the Gaussian process

$$\begin{pmatrix} 1 & -1 & 1 & 4 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 6 & 13 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & -1 & 1 & 4 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This shows that $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ are linearly independent, and $\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4$. The correct answer is alternative **C**.

Question 3.
We compute the eigenvalues of A by solving the characteristic equations $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} 7-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & 7-\lambda \end{vmatrix} = 0$$

We compute the determinant by cofactor expansion along the first row, which gives

$$(2-\lambda) \cdot \begin{vmatrix} 7-\lambda & 2 \\ 2 & 7-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 14\lambda + 45)$$

Since $\lambda^2 - 14\lambda + 45 = 0$ has solutions $\lambda = 5$ and $\lambda = 9$, there are three distinct eigenvalues $\lambda = 2, 5, 9$ of multiplicity one. The correct answer is alternative **A**.

Question 4.
We compute the minor $M_{12,24}$ of A:

$$M_{12,24} = \begin{vmatrix} 6 & -t \\ t & 1 \end{vmatrix} = 6 + t^2$$

This minor is non-zero for all values of t, hence $\text{rk}(A) = 2$ for all values of t. The correct answer is alternative **D**.

Question 5.
Since the Markov chain is regular, we find all eigenvectors with $\lambda = 1$. We solve the linear system $(A - I)\mathbf{v} = \mathbf{0}$ by the Gaussian process

$$\begin{pmatrix} -0.6 & 0.2 & 0.2 \\ 0.4 & -0.4 & 0.1 \\ 0.2 & 0.2 & -0.3 \end{pmatrix} \quad \to \quad \begin{pmatrix} 2 & 2 & -3 \\ 4 & -4 & 1 \\ -6 & 2 & 2 \end{pmatrix} \quad \to \quad \begin{pmatrix} 2 & 2 & -3 \\ 0 & -8 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence z is free, $-8y + 7z = 0$ gives $y = 7z/8$, and $2x + 2y - 3z = 0$ gives $2x = -14z/8 + 3z = 10z/8$, or $x = 5z/8$. The unique state vector in E_1 is therefore given by $v_1 = 5/20 = 0.25$, $v_2 = 7/20 = 0.35$ and $v_3 = 8/20 = 0.40$ (for $z = 8/20$). The correct answer is alternative **D**.

Question 6.

The symmetric matrix of the quadratic form $f(x,y,z) = 2xy - 2x^2 - y^2 + 2xz - z^2$ is given by

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

The leading principal minors are $D_1 = -2$, $D_2 = 2 - 1 = 1$, $D_3 = -1(2-1) + 1(1) = 0$. Since A has $\operatorname{rk} A = 2$, it follows by the reduced rank criterion that A and f are negative semidefinite, but not negative definite. We could also check this by computing all principal minors. The correct answer is alternative **B**.

Question 7.

The function $f(x,y,z) = x^4 + y^4 + z^4 - 4xy$ has first order partial derivatives and first order conditions given by

$$f'_x = 4x^3 - 4y = 0, \quad f'_y = 4y^3 - 4x = 0, \quad f'_z = 4z^3 = 0$$

The stationary points are given by $z = 0$, $x^3 = y$, and $x = y^3$. This gives $y = x^3 = (y^3)^3 = y^9$, or $y(1 - y^8) = 0$. Since $y^8 = 1$ gives $y = \pm 1$, there are three stationary point given by $y = 0$, $y = 1$ and $y = -1$; the stationary points are $(0,0,0)$, $(1,1,0)$, and $(-1,-1,0)$. The Hessian matrix is

$$H(f) = \begin{pmatrix} 12x^2 & -4 & 0 \\ -4 & 12y^2 & 0 \\ 0 & 0 & 12z^2 \end{pmatrix}$$

which gives

$$H(f)(0,0,0) = \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H(f)(\pm 1, \pm 1, 0) = \begin{pmatrix} 12 & -4 & 0 \\ -4 & 12 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We have that $H(f)(0,0,0)$ is indefinite, since $D_2 = -16 < 0$, hence $(0,0,0)$ is a saddle point. We also have that $H(f)(\pm 1, \pm 1, 0)$ is positive semi-definite (but not positive definite), since the principal minors are given by $D_1 = 12$, $D_2 = 144 - 16 = 128 > 0$ and $D_3 = 0$ and we can apply the reduced rank criterion since the matrix has rank two. Hence the second derivative test is inconclusive. The stationary points $(1,1,0)$ and $(-1,-1,0)$ are in fact local minimum points: To see this, note that $f(x,y) = x^4 + y^4 - 4xy$ has local minimum points $(1,1)$, $(-1,-1)$ by the second derivative test and the results above. Moreover, $f(1,1,0) = -2$ and $f(1,1,z) = -2 + z^4 \geq -2$. The correct answer is alternative **B**.

Question 8.

Since $\mathbf{v}_3 = -3\mathbf{v}_1 + \mathbf{v}_2$, and $\{\mathbf{v}_1, \mathbf{v}_2\}$ clearly are linearly independent, it follows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a base of $\text{Null}(A)$. Hence $\dim \text{Null}(A) = 2$ equals the number of free variables, and $\operatorname{rk} A = 4 - 2 = 2$. The correct answer is alternative **C**.