EVALUATION GUIDELINES - Multiple choice

GRA 60352 Mathematics

Department of Economics

Start date:	11.10.2019	Time 15:00
Finish date:	11.10.2019	Time 16:00

For more information about formalities, see examination paper.

Solutions	Midterm exam in GRA 6035 Mathematics
Date	October 11th, 2019 at 1500 - 1600

Correct answers: C-C-D-D A-B-A-D

Question 1.
We find the pivot positions in A, given by the Gaussian process

$$\begin{pmatrix} 1 & 4 & 3 & 5 & 7 \\ 0 & 0 & 0 & 5 & 13 \\ 0 & 4 & 0 & 1 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 3 & 5 & 7 \\ 0 & 4 & 0 & 1 & 5 \\ 0 & 0 & 0 & 5 & 13 \end{pmatrix}$$

Hence there is one degree of freedom. The correct answer is alternative **C**.

Question 2.
We find the pivot positions in A, given by the Gaussian process

$$\begin{pmatrix} 1 & 4 & 3 & 5 & 7 \\ 0 & 0 & 0 & 5 & 13 \\ 0 & 4 & 0 & 1 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 3 & 5 & 7 \\ 0 & 4 & 0 & 1 & 5 \\ 0 & 0 & 0 & 5 & 13 \end{pmatrix}$$

This shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ are linearly independent, and $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$. The correct answer is alternative **C**.

Question 3.
We compute the eigenvalues of A by solving the characteristic equations $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} 7-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ -2 & 0 & 7-\lambda \end{vmatrix} = 0$$

We compute the determinant by cofactor expansion along the first row, which gives

$$(2-\lambda) \cdot \begin{vmatrix} 7-\lambda & 2 \\ -2 & 7-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 14\lambda + 53)$$

Since $\lambda^2 - 14\lambda + 53 = 0$ has no (real) solutions, there is only one eigenvalue $\lambda = 2$ of multiplicity one. The correct answer is alternative **D**.

Question 4.
We compute the minor $M_{12,14}$ of A:

$$M_{12,14} = \begin{vmatrix} t & 1 \\ -1 & t \end{vmatrix} = t^2 + 1$$

This minor is non-zero for all values of t, hence $\text{rk}(A) = 2$ for all values of t. The correct answer is alternative **D**.

Question 5.
Since the Markov chain is regular, we find all eigenvectors with $\lambda = 1$. We solve the linear system $(A - I)\mathbf{v} = \mathbf{0}$ by the Gaussian process

$$\begin{pmatrix} -0.6 & 0.2 & 0.1 \\ 0.4 & -0.4 & 0.1 \\ 0.2 & 0.2 & -0.2 \end{pmatrix} \to \begin{pmatrix} 2 & 2 & -2 \\ 4 & -4 & 1 \\ -6 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 2 & -2 \\ 0 & -8 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence z is free, $-8y + 5z = 0$ gives $y = 5z/8$, and $2x + 2y - 2z = 0$ gives $x = -5z/8 + z = 3z/8$. The unique state vector in E_1 are therefore given by $v_1 = 3/16$, $v_2 = 5/16$ and $v_3 = 8/16$ (for $z = 1/2$). The correct answer is alternative **A**.

Question 6.

The symmetric matrix of the quadratic form $f(x, y, z) = 2xy - x^2 - 2y^2 + 2yz - z^2$ is given by

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

The leading principal minors are $D_1 = -1$, $D_2 = 2 - 1 = 1$, $D_3 = -1(2-1) - 1(-1) = 0$. Since A has $\operatorname{rk} A = 2$, it follows by the reduced rank criterion that A and f are negative semidefinite, but not negative definite. We could also check this by computing all principal minors. The correct answer is alternative **B**.

Question 7.

The function $f(x, y, z) = x^3 + y^3 + z^3 - 3xz$ has first order partial derivatives and first order conditions given by

$$f'_x = 3x^2 - 3z = 0, \quad f'_y = 3y^2 = 0, \quad f'_z = 3z^2 - 3x = 0$$

The stationary points are given by $z = x^2$, $y = 0$, and $x = z^2$. This gives $z = x^2 = (z^2)^2 = z^4$, or $z(1 - z^3) = 0$. There are two stationary point given by $z = 0$ and $z = 1$, which are $(0, 0, 0)$ and $(1, 0, 1)$. The Hessian matrix is

$$H(f) = \begin{pmatrix} 6x & 0 & -3 \\ 0 & 6y & 0 \\ -3 & 0 & 6z \end{pmatrix} \quad \Rightarrow \quad H(f)(0,0,0) = \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{pmatrix}, \quad H(f)(1,0,1) = \begin{pmatrix} 6 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 6 \end{pmatrix}$$

We have that $H(f)(0, 0, 0)$ is indefinite, since the principal 2-minor $M_{13,13} = -9 < 0$, so $(0, 0, 0)$ is a saddle point. We also have that $H(f)(1, 0, 1)$ is positive semi-definite (but not positive definite), since the principal minors are given by $\Delta_1 = 6, 0, 6$, $\Delta_2 = 0, 0, 27$ and $\Delta_3 = 0$. Hence the second derivative test is inconclusive. The stationary point $(1, 0, 1)$ is a saddle point, since $f(1, 0, 1) = -1$ and $f(1, y, 1) = -1 + y^3$ can both be less than and more than -1 for values of y close to zero (a small negative value for y gives $f < -1$ and a small positive value for y gives $f > -1$). The correct answer is alternative **A**.

Question 8.

Since $\dim \operatorname{Null}(A) = 2$ equals the number of free variables, we have that $A\mathbf{x} = \mathbf{0}$ has two degrees of freedom. The correct answer is alternative **D**.