Solutions	Midterm exam in GRA 6035 Mathematics
Date	January 5th, 2018 at 1500 - 1600

Correct answers: A-D-D-A-A-D-B-C

Question 1.

Since $\operatorname{rk} A = n$, where n is the number of columns in A, and $\mathbf{b} = \mathbf{0}$, we also have $\operatorname{rk}(A|\mathbf{b}) = n$, and the linear system is consistent with $n - n = 0$ degrees of freedom, that is a unique solution. The correct answer is alternative **A**.

Question 2.

We form the matrix with the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as columns, and compute its determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & t \end{vmatrix} = 1(2t - 12) - 1(t - 3) + 1(4 - 2) = t - 7$$

This shows that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent when $t = 7$, and linearly independent otherwise. The correct answer is alternative **D**.

Question 3.

We compute the three 2-minors (maximal minors) of the matrix A:

$$M_{12,12} = t^2 - 1, \quad M_{12,13} = 0, \quad M_{12,23} = 1 - t^2$$

This means that $\operatorname{rk}(A) < 2$ when $t^2 = 1$, or when all maximal minors vanish, and that $\operatorname{rk}(A) = 2$ otherwise. Therefore, $\operatorname{rk}(A) = 2$ for $t \neq 1, -1$. The correct answer is alternative **D**.

Question 4.

We compute the eigenvalues of A by solving the characteristic equations $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} -1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

We compute the determinant by cofactor expansion along the first row, which gives

$$(-1-\lambda) \cdot \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (-1-\lambda)(\lambda^2 - 4\lambda + 3) = (-1-\lambda)(\lambda - 1)(\lambda - 3) = 0$$

Therefore, $\lambda = -1$, $\lambda = 1$ and $\lambda = 3$ are eigenvalues of multiplicity one. The correct answer is alternative **A**.

Question 5.

The eigenvalues are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$. Since $\lambda = 1$ is the only eigenvalue of multiplicity $m = 2 > 1$, we consider the eigenspace E_1 of solutions of the linear system $(A - \lambda I) \cdot \mathbf{x} = \mathbf{0}$ for $\lambda = 1$. In matrix form, it can be written

$$\begin{pmatrix} 0 & s & -s^2 \\ 0 & -1 & s \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We use Gaussian elimination, and obtain the echelon form (of the coefficient matrix) given by

$$\begin{pmatrix} 0 & s & -s^2 \\ 0 & -1 & s \\ 0 & 0 & 0 \end{pmatrix} \quad \to \quad \begin{pmatrix} 0 & -1 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that for all values of s, the only pivot position is in the second column, and both x and z are free. The matrix A is therefore diagonalizable for all values of s. The correct answer is alternative **A**.

Question 6.

The symmetric matrix of the quadratic form $f(x, y, z) = -x^2 + 4xy + 2xz - 4y^2 + 4yz$ is given by

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

The leading principal minors are $D_1 = -1$, $D_2 = 0$ and $D_3 = |A| = 1(4+4) - 2(-2-2) = 16$ (we use cofactor expansion along the last row). Since $D_1 < 0$ and $D_3 > 0$, we conclude that f is indefinite. The correct answer is alternative **D**.

Question 7.

The function $f(x, y, z) = 1 - x^4 - 2x^2 - 4xz - y^2 - z^4 - 2z^2$ has a stationary point in $(x, y, z) = (0, 0, 0)$ since the first order partial derivatives

$$f'_x = -4x^3 - 4x - 4z, \quad f'_y = -2y, \quad f'_z = -4x - 4z^3 - 4z$$

are zero at this point. The Hessian matrix of f is given by

$$H(f) = \begin{pmatrix} -12x^2 - 4 & 0 & -4 \\ 0 & -2 & 0 \\ -4 & 0 & -12z^2 - 4 \end{pmatrix}$$

The leading principal minors are $D_1 = -12x^2 - 4$, $D_2 = -2D_1$, and $D_3 = -2(144x^2z^2 + 48x^2 + 48z^2)$. We see that $D_1 < 0$, $D_2 > 0$ and $D_3 \le 0$ for all points (x, y, z). At points (x, y, z) with $D_3 < 0$, it is clear that $H(f)$ is negative definite. At points (x, y, z) with $D_3 = 0$, the matrix $H(f)$ has rank two, and therefore $H(f)$ is negative semidefinite. We conclude that f is concave, and $(x, y, z) = (0, 0, 0)$ is a global maximum for f. The correct answer is alternative **B**.

Question 8.

Eigenvectors of A for $\lambda = 1$ are given by the linear system $(A - I)\mathbf{x} = \mathbf{0}$, where

$$A - I = \begin{pmatrix} -0.23 & 0.17 \\ 0.23 & -0.17 \end{pmatrix}$$

Therefore, we see that $-0.23x + 0.17y = 0$ and $x = 17y/23$. The eigenvectors are therefore given by

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = y/23 \begin{pmatrix} 17 \\ 23 \end{pmatrix}$$

The condition $x + y = 1$ for a state vector gives $y/23 = 1/(17 + 23)$, or $y = 23/40$ and the market share of Firm A in the long run is $x = 17/40 = 42.5\%$. The correct answer is alternative **C**.