Solutions	Midterm exam in GRA 6035 Mathematics
Date	May 18th, 2017 at 0900 - 1000

Correct answers: B-D-C-A-A-D-C-B

Question 1.

Since $(A|\mathbf{b})$ is a 5×5-matrix of rank five, there is a pivot postion in the last column, and the system is inconsistent. The correct answer is alternative **B**.

Question 2.

We form the matrix with the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as columns, and compute its determinant

$$\begin{vmatrix} 4 & -4 & 2t \\ t & 3 & 0 \\ 3 & 1 & 1 \end{vmatrix} = 2t(t-9) + 1(12+4t) = 2t^2 - 14t + 12 = 2(t^2 - 7t + 6) = 2(t-6)(t-1)$$

This shows that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent when $t = 1, 6$, and linearly independent otherwise. The correct answer is alternative **D**.

Question 3.

We use elementary row operations to find an echelon form of the coefficient matrix A:

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & t \\ 3 & 4 & t+1 \\ 2 & -1 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 \\ 0 & -5 & t-2 \\ 0 & -5 & t-2 \\ 0 & -7 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 \\ 0 & -7 & 7 \\ 0 & -5 & t-2 \\ 0 & -5 & t-2 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & t-7 \\ 0 & 0 & 0 \end{pmatrix}$$

This means that there is one free variables if $t = 7$, and zero degrees of freedom if $t \neq 7$. The correct answer is alternative **C**.

Question 4.

We have that $\lambda_1 + \lambda_2 + \lambda_3 = \text{tr}A = 40$ and that $\lambda_1\lambda_2\lambda_3 = \det(A) = 6(17^2 - 16 \cdot 9) = 198$. We can also compute $\lambda_1 = 6$, $\lambda_2 = 5$ and $\lambda_3 = 29$ explicitly to see this. The correct answer is alternative **A**.

Question 5.

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = s$ and $\lambda_3 = s-1$. When $s \neq 1, 2$, there are three distinct eigenvalues, and A is diagonalizable. For $s = 1$, the eigenvalues are $1, 1, 2$, and the eigenspace $(A - I)\mathbf{x} = \mathbf{0}$ for $\lambda = 1$ has two free variables while the multiplicity of $\lambda = 1$ is two. When $s = 2$, the eigenvalues are $1, 2, 1$, and the eigenspace $(A - I)\mathbf{x} = \mathbf{0}$ for $\lambda = 1$ has two free variables while the multiplicity of $\lambda = 1$ is two. In both cases, A is diagonalizable. The correct answer is alternative **A**.

Question 6.

Eigenvectors for $\lambda = 1$ are given by the linear system $(A - I)\mathbf{x} = \mathbf{0}$, where

$$A - I = \begin{pmatrix} -0.28 & 0.28 \\ 0.28 & -0.28 \end{pmatrix}$$

Therefore, we see that $x = 1$ and $y = 1$ gives one eigenvector, and all others are multiple of this one. Multiplication by $1/2$ gives the state vector with $x = 1/2$ and $y = 1/2$. The correct answer is alternative **D**.

Question 7.

The symmetric matrix of the quadratic form $f(x, y, z) = 3x^2 - 8xy - 4xz + 5y^2 - 4yz + 8z^2$ is given by

$$A = \begin{pmatrix} 3 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{pmatrix}$$

The first leading principal minors are $D_1 = 3$ and $D_2 = 15 - 16 = -1$. Since $D_2 < 0$, the quadratic form is indefinite. The correct answer is alternative **C**.

Question 8.

The function $f(x, y) = \sqrt{x^2 + y^2 + 3} = \sqrt{u}$ with $u = x^2 + y^2 + 3$ has Hessian matrix

$$H(f) = \begin{pmatrix} \frac{y^2+3}{u\sqrt{u}} & -\frac{xy}{u\sqrt{u}} \\ -\frac{xy}{u\sqrt{u}} & \frac{x^2+3}{u\sqrt{u}} \end{pmatrix}$$

Since $u = x^2 + y^2 + 3 > 0$ for all (x, y), we have that $D_1 = \frac{y^2+3}{u\sqrt{u}} > 0$ and that

$$D_2 = \frac{y^2 + 3}{u\sqrt{u}} \cdot \frac{x^2 + 3}{u\sqrt{u}} - \left(-\frac{xy}{u\sqrt{u}}\right)^2 = \frac{3x^2 + 3y^2 + 9}{u^3} > 0$$

for all (x, y). It follows that f is convex but not concave. The correct answer is alternative **B**.