Solutions	Final exam in GRA 6035 Mathematics
Date	January 9th 2026 at 0900 - 1400

Question 1.

(a) The differential equation $y' - 3y = 12$ is a linear first order differential equation, and it has characteristic equation $r - 3 = 0$. This gives $r = 3$ and $y_h = Ce^{3t}$. Since the right-hand side is constant, we try $y(t) = A$ to find a particular solution. This gives $y' = 0$, and $0 - 3A = 12$, or $y_p = A = -4$ Hence the general solution is $y = C \cdot e^{3t} - 4$.

(b) We find an echelon form of A using Gaussian elimination:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 3 \\ 3 & 10 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Since the echelon form has three pivot positions, we find that $\operatorname{rk} A = 3$. Alternatively, we could compute $|A| = 1(0) - 3(1) + 2(2) = 1$, and conclude that $\operatorname{rk} A = 3$ since $|A| \neq 0$.

(c) The quadratic function can be written in the form $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + B\mathbf{x} + 3$, with stationary points given by $f'(\mathbf{x}) = 2A\mathbf{x} + B^T = \mathbf{0}$, or $A\mathbf{x} = -B^T/2$. The symmetric matrix A and the augmented matrix of the first order conditions $A\mathbf{x} = -B^T/2$ are given by

$$A = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} \quad \text{and} \quad \left(\begin{array}{cc|c} 5 & -3 & 2 \\ -3 & 2 & -1 \end{array}\right)$$

We see that A is positive definite since $D_1 = 5$ and $D_2 = 1$. Hence f is convex, and there is a unique stationary point since $|A| \neq 0$. We note that $(1, 1)$ is a solution of the linear system with augmented matrix given above (the FOC's), therefore it is the unique stationary point of f, and the minimum point of f since f is convex. We conclude that $f_{\min} = f(1, 1) = 2$.

(d) We use that for any vector $\mathbf{u}$, we have that $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$, and compute

$$||\mathbf{v} - \mathbf{w}||^2 = (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - 2(\mathbf{v} \cdot \mathbf{w}) + \mathbf{w} \cdot \mathbf{w} = 2^2 - 2(2) + 3^2 = 9$$

This implies that $||\mathbf{v} - \mathbf{w}|| = \sqrt{9} = 3$.

Question 2.

(a) We know that $\dim \operatorname{Col}(A) = \operatorname{rk} A$ is the number of pivot positions in A, and we find an echelon form of A to determine the rank:

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Since there are three pivot positions, it follows that $\operatorname{rk} A = 3$. We also know that the column vectors corresponding to pivot positions form a base of $\operatorname{Col}(A)$. In this case, this is the first, second and fourth column of A, and a base of $\operatorname{Col}(A)$ is given by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ with

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

(b) We use the echelon form of A given above to find a base of the nullspace of A: We see that x_3 is a free variable, and back substitution gives that $-x_4 = 0$, or $x_4 = 0$, that $-x_2 + x_3 - 0 = 0$, or $x_2 = x_3$, and that $x_1 + x_3 + 2(0) = 0$, or $x_1 = -x_3$. Hence the nullspace is given by

$$\mathbf{x} = \begin{pmatrix} -x_3 \\ x_3 \\ x_3 \\ 0 \end{pmatrix} = x_3 \cdot \mathbf{w}, \quad \text{where } \mathbf{w} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \text{ is a base of } \operatorname{Null}(A)$$

This means that $-\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, and it is possible to express any of these three vectors in terms of the other two. For instance, we have that $\mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2$.

(c) We have that $|B| = 0$ since the last row is the sum of the first two rows, and compute $\text{tr}(B) = 1 + 0 + 1 = 2$ and $c_2 = -1 + (-1) + 1 = -1$. This means that the characteristic equation of B is

$$-\lambda^3 + 2\lambda^2 - (-1)\lambda + 0 = -\lambda(\lambda^2 - 2\lambda - 1) = 0$$

Since $\lambda^2 - 2\lambda - 1 = 0$ has solutions $\lambda = (2 \pm \sqrt{4+4})/2 = 1 \pm \sqrt{8}/2 = 1 \pm \sqrt{2}$, the matrix has the three distinct eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1 + \sqrt{2} \approx 2.41$ and $\lambda_3 = 1 - \sqrt{2} \approx -0.41$. Since B has three distinct eigenvalues, it is diagonalizable.

Question 3.

(a) The quadratic form u can be written in matrix form as $u(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, where A is the symmetric matrix

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

The leading principal minors of A are $D_1 = 4$, $D_2 = 4-4 = 0$ and $D_3 = 1(3)-2(6)+1(0) = -9$, and u is indefinite since D_1 and D_3 have opposite signs.

(b) The derivative is $f'(u) = (e^u + 4e^{-u})' = e^u + 4e^{-u} \cdot (-1) = e^u - 4e^{-u}$. To find the stationary points of f, we solve the first order condition $f'(u) = 0$:

$$f'(u) = e^u - \frac{4}{e^u} = 0 \quad \Rightarrow \quad (e^u)^2 - 4 = 0 \quad \Rightarrow \quad (e^u)^2 = 4 \quad \Rightarrow \quad e^u = \pm\sqrt{4} = \pm 2$$

Since e^u is positive, this implies that $e^u = 2$, and therefore $u = \ln(2) \approx 0.69$ is the unique stationary point of f.

(c) From (a) we know that the quadratic form u is indefinite, hence it has no max or min (its stationary points are saddle points), and the range of u is $V_u = (-\infty, \infty)$. In fact, we have that $u(x, 0, 0) = 4x^2 \to \infty$ when $x \to \infty$, and $u(0, -z, z) = (-z)^2 + 4(-z)z + z^2 = -2z^2 \to -\infty$ when $z \to \infty$. This means that the values of the composite function are the values of the function $f(u) = e^u + 4e^{-u}$ defined for all values of u. The stationary point $u = \ln(2)$ is a global minimum point of f, since we have that

$$f''(u) = (e^u - 4e^{-u})' = e^u - 4e^{-u} \cdot (-1) = e^u + 4e^{-u} > 0$$

which means that f is a convex function. Alternatively, we could use a sign diagram for $f'(u)$ to determine that the stationary point is a global minimum point of f. This means that $f_{\min} = f(\ln 2) = 2 + 4(1/2) = 4$, and f has no maximum value since $f(u) = e^u + 4e^{-u} \to \infty$ when $u \to \infty$. We conclude that the range of f is $V_f = [4, \infty)$.

Question 4.

(a) The difference equation is second order linear, with general solution $y_t = y_t^h + y_t^p$. The characteristic equation of the homogeneous equation is $r^2 - 5r + 6 = 0$, with roots $r = 2$ and $r = 3$, and it follows that $y_t^h = A \cdot 2^t + B \cdot 3^t$ is the homogeneous solution. To find the particular solution of $y_{t+2} - 5y_{t+1} + 6y_t = 2$, we try the constant $y_t = C$ since 2 is a constant. This gives $C - 5C + 6C = 2$, which means that $2C = 2$, or $C = 1$, and that $y_t^p = 1$. The general solution of the inhomogeneous difference equation is therefore

$$y_t = y_t^h + y_t^p = A \cdot 2^t + B \cdot 3^t + 1$$

(b) We write the system in the form $\mathbf{y}' = A\mathbf{y}$. To solve the system, we find the eigenvalues and eigenvectors of A: We have that $\text{tr}(A) = 1+4-2 = 3$ and $\det(A) = 0$ since the third row is the sum of the two first rows. Moreover, $c_2 = 0 + 4 - 2 = 2$. This implies that the characteristic equation is

$$-\lambda^3 + 3\lambda^2 - 2\lambda + 0 = -\lambda(\lambda^2 - 3\lambda + 2) = -\lambda(\lambda - 1)(\lambda - 2) = 0$$

The eigenvalues of A are therefore $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$. Alternatively, we can computing $|A - \lambda I|$ using cofactor expansion. Since all eigenvalues have multiplicity one, there is a base

$\mathbf{v}_i$ for E_{λ_i} which we can find using Gaussian elimination:

$$E_0: \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & -2 \\ 3 & 6 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \qquad E_1: \begin{pmatrix} 0 & 2 & 0 \\ 2 & 3 & -2 \\ 3 & 6 & -3 \end{pmatrix} \to \begin{pmatrix} 3 & 6 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_2: \begin{pmatrix} -1 & 2 & 0 \\ 2 & 2 & -2 \\ 3 & 6 & -4 \end{pmatrix} \to \begin{pmatrix} -1 & 2 & 0 \\ 0 & 6 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Using back substitution, we see that we may choose the base vectors as $\mathbf{v}_1 = (-2, 1, 0)$, $\mathbf{v}_2 = (1, 0, 1)$ and $\mathbf{v}_3 = (2, 1, 3)$. The system therefore has general solution

$$\mathbf{y} = C_1\mathbf{v}_1 e^{\lambda_1 t} + C_2\mathbf{v}_2 e^{\lambda_2 t} + C_3\mathbf{v}_3 e^{\lambda_3 t} = C_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot e^t + C_3 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \cdot e^{2t}$$

(c) The differential equation $t^2 y' = (1-y)^2$ is separable since it can be written in the form

$$\frac{1}{(1-y)^2}\, y' = \frac{1}{t^2} \quad \Leftrightarrow \quad (1-y)^{-2}\, y' = t^{-2}$$

Integration with respect to t on both sides is made easier by writing the separated form in terms of powers, and the power rule of integration gives

$$\int (1-y)^{-2}\, \mathrm{d}y = \int t^{-2}\, \mathrm{d}t \quad \Rightarrow \quad \frac{1}{1-y} = -\frac{1}{t} + C$$

when we take into account that $(1-y)' = -1$. The initial condition $y(1) = 0$ gives $1 = -1 + C$, or $C = 2$, and we use this to find the solution in explicit form:

$$\frac{1}{1-y} = -\frac{1}{t} + 2 = \frac{-1+2t}{t} \;\Rightarrow\; 1 - y = \frac{t}{2t-1} \;\Rightarrow\; y = 1 - \frac{t}{2t-1} = \frac{2t-1-t}{2t-1} = \frac{t-1}{2t-1}$$

Question 5.

(a) The Langrangian of this problem is given by $\mathcal{L} = x+2y+4z+w+2xw-\lambda(3x^2+y^2+4z^2+w^2-a)$, and the first order conditions can be written as

$$\mathcal{L}'_x = 1 + 2w - \lambda(6x) = 0$$
$$\mathcal{L}'_y = 2 - \lambda(2y) = 0$$
$$\mathcal{L}'_z = 4 - \lambda(8z) = 0$$
$$\mathcal{L}'_w = 1 + 2x - \lambda(2w) = 0$$

and the constraint is $3x^2 + y^2 + 4z^2 + w^2 = a$. When $\lambda = 0$, the second FOC gives $2 = 0$, which is impossible. Therefore, there are no candidate points with $\lambda = 0$ for any value of a.

(b) When $\lambda = 1$, the two middle FOC's give $y = 1$ and $z = 1/2$, and the first and last FOC's give $6x - 2w = 1$ and $-2x + 2w = 1$. Adding these equations, we find that $4x = 2$, or $x = 1/2$, and if we substitute this into the first equation, we get $3 - 2w = 1$, or $w = 1$. We therefore find one candidate point that satisfies the FOC with $\lambda = 1$, and when we substitute this point into the constraint, we get $3/4 + 1 + 1 + 1 = a$, or $a = 15/4$. We conclude that there are candidate points with $\lambda = 1$ when $a = 15/4 = 3.75$, and in this case the unique candidate point is $(x, y, z, w; \lambda) = (1/2, 1, 1/2, 1; 1)$.

(c) When $a = 15/4$, we have the candidate point $(1/2, 1, 1/2, 1; 1)$ from (b), and we use the SOC to prove that this is a maximum point: When $h = \mathcal{L}(x, y, z, w; 1)$, we have that $H(h) = 2A$ where A is given by

$$A = \begin{pmatrix} -3 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

with leading principal minors $D_1 = -3$, $D_2 = 3$, $D_3 = -12$, and $D_4 = -1(4) - 1(-12) = 8$. It follows that A is negative definite (and therefore negative semidefinite), and this means that h is a concave function. By the SOC, the candidate point is a maximum point, with maximum

value $f(1/2, 1, 1/2, 1) = 1/2 + 2 + 2 + 1 + 1 = 6.5$. We conclude that $f^*(15/4) = 13/2 = 6.5$, and we use the envelope theorem to estimate $f^*(4)$:

$$f^*(4) \approx f^*(3.75) + \Delta a \cdot \frac{\mathrm{d}f^*(a)}{\mathrm{d}a} = 6.5 + (4 - 3.75) \cdot 1 = 6.75$$

since $\mathcal{L}'_a = \lambda$ and $\lambda^*(a) = 1$ when $a = 15/4$.