Solutions	Final exam in GRA 6035 Mathematics
Date	January 10th 2025 at 0900 - 1400

Question 1.

(a) We use superposition to solve the first order difference equation $y_{t+1} - 2y_t = 200$. To find the homogeneous solution, we solve the characteristic equation $r - 2 = 0$, which gives $r = 2$ and $y_t^h = C \cdot 2^t$. To find a particular solution, we set $y_t = A$ since 200 is constant. This gives $A - 2A = 200$, or $A = -200$. Therefore the particular solution is $y_t^p = -200$, and the general solution of the difference equation is $y_t = y_t^h + y_t^p = C \cdot 2^t - 200$.

(b) The quadratic form $f(x, y, z) = x^2 - xz + yz$ has symmetric matrix

$$A = \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 1/2 \\ -1/2 & 1/2 & 0 \end{pmatrix}$$

The leading principal minors $D_1 = 1$, $D_2 = 0$, and $D_3 = 1(-1/4) - 1/2(0) = -1/4$. Since D_1 is positive and D_3 is negative, the quadratic form is indefinite.

(c) The autonomous differential equation $y' = y(1 - y^2)$ has equilibrium states given by $y' = 0$, or $F(y) = y(1 - y^2) = y(1 - y)(1 + y) = 0$, and this gives $y_e = -1, 0, 1$. To determine their stability, we compute $F'(y) = (y - y^3)' = 1 - 3y^2$ at the equilibrium states y_e. Since $F'(\pm 1) = -2 < 0$ and $F'(0) = 1 > 0$, we conclude that the stable equilibrium states are $y_e = 1$ and $y_e = -1$.

(d) The matrix A is upper triangular, and therefore has three eigenvalues (counted with multiplicity) equal to the diagonal entries $\lambda = 1$ and $\lambda = 2$ (with multiplicity two). Therefore A is diagonalizable if and only if $\dim E_2 = 2$. Since $\dim E_2 = 3 - \text{rk}(A - 2I) = 2$ if and only if $\text{rk}(A - 2I) = 1$, we consider

$$A - 2I = \begin{pmatrix} -1 & s & 1 \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix}$$

This is an echelon form, and it has rank one if and only if $s = 0$. We conclude that A is diagonalizable if and only if $s = 0$.

Question 2.

(a) The trace of A is $\text{tr}(A) = 2 + 2 + 2 = 6$, and we find the determinant of A using cofactor expansion along the first row:

$$\det(A) = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 2(4 - 1) - 1(2 - 1) + (-1)(-1 + 2) = 6 - 1 - 1 = 4$$

This gives $\text{tr}(A) = 6$ and $\det(A) = 4$.

(b) To find the characteristic equation of A, we use that $\text{tr}(A) = 6$ and $\det(A) = 4$, and compute

$$c_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 + 3 + 3 = 9$$

It follows that the characteristic equation is $-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$. Alternatively, we could find the characteristic equation of A by computing $|A - \lambda I|$ using cofactor expansion:

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = (2 - \lambda)((2 - \lambda)^2 - 1) - 1(2 - \lambda - 1) + (-1)(-1 + 2 - \lambda)$$

$$= (2 - \lambda)(\lambda^2 - 4\lambda + 3) - 2(1 - \lambda) = -\lambda^3 + 6\lambda^2 - 9\lambda + 4$$

We see that when we substitute $\lambda = 1$, $\lambda = 2$, and $\lambda = 4$ in the expression on the left-hand side of the characteristic equation, we get

$$-1^3 + 6 \cdot 1^2 - 9 \cdot 1 + 4 = 0, \quad -2^3 + 6 \cdot 2^2 - 9 \cdot 2 + 4 = 2, \quad -4^3 + 6 \cdot 4^2 - 9 \cdot 4 + 4 = 0$$

We conclude that $\lambda = 1$ and $\lambda = 4$ are eigenvalues of A, but $\lambda = 2$ is not an eigenvalue of A.

(c) We claim that the symmetric matrix A is positive definite: One way to see this is by computing all eigenvalues of A: Since $\lambda = 1$ and $\lambda = 4$ are eigenvalues by Question 2(b), the third eigenvalue is given by $1 + 4 + \lambda = \text{tr}(A) = 6$, or $\lambda = 1$. Since A has three positive eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 4$, it is positive definite. Alternatively, we could compute the leading principal minors of A: Since $D_1 = 2$, $D_2 = 4 - 1 = 3$ and $D_3 = |A| = 4$ are positive, A is positive definite. Using that f is a quadratic form, with $(0, 0, 0)$ as a stationary point, and that f is positive definite and therefore convex, it follows that the origin is a global minimum point with value $f(0, 0, 0) = 0$. The range of f is $V_f = [0, \infty)$ since $f(x, 0, 0) = 2x^2 \to \infty$ as $x \to \infty$.

(d) Since $f(x, 0, 0) = 2x^2 \geq 8$ for $x^2 \geq 4$, we have that $(x, 0, 0)$ is in D for $x \geq 2$ and $x \leq -2$. This means that D is not bounded, and therefore D is not compact.

Question 3.

(a) The quadratic form $f(x, y, z) = xy - xz - yz$ has symmetric matrix

$$A = \begin{pmatrix} 0 & 1/2 & -1/2 \\ 1/2 & 0 & -1/2 \\ -1/2 & -1/2 & 0 \end{pmatrix}$$

Since the leading principal minor $D_2 = -1/4 < 0$, the quadratic form is indefinite, and this implies that any stationary points of u is a saddle point. Hence u has no (local or global) max or min, and the range of u is given by $V_u = \mathbb{R} = (-\infty, \infty)$. Alternatively, we could use the fact that $f(x, 1, 0) = x \to \pm\infty$ when $x \to \pm\infty$ to show this.

(b) Since the range of u is $\mathbb{R} = (-\infty, \infty)$, we consider the outer function $f(u) = e^u + e^{-u}$ as a function in one variable u with domain of definition $D_f = \mathbb{R} = (-\infty, \infty)$. Its first and second derivatives are given by

$$f'(u) = e^u + e^{-u} \cdot (-1) = e^u - e^{-u}, \quad f''(u) = e^u - e^{-u} \cdot (-1) = e^u + e^{-u}$$

We find the stationary points by solving the first order condition $f'(u) = e^u - e^{-u} = 0$, which gives $e^u = e^{-u}$, or $u = -u$. Hence $2u = 0$, and $u = 0$ is the stationary point of f. Since $f''(u) > 0$ for all u, f is a convex function, and $u = 0$ is the global minimum point of f. Clearly f has no maximum since $f(u) = e^u + e^{-u} \to \infty$ when $u \to \pm\infty$. We conclude that f has a minimum value $f_{\min} = f(0) = e^0 + e^{-0} = 2$, and no maximum value.

Question 4.

(a) We solve the differential equation $y'' - 7y' - 8y = 1 - 6t - 8t^2$ using superposition. Since the characteristic equation is $r^2 - 7r - 8 = (r - 8)(r + 1) = 0$, with characteristic roots $r = 8$ and $r = -1$, the homogeneous solution is $y_h = C_1 e^{8t} + C_2 e^{-t}$. To find a particular solution, we substitute $y = At^2 + Bt + C$ into the differential equation. This gives $y' = 2At + B$ and $y'' = 2A$, and the left-hand side of the differential equation becomes

$$(2A) - 7(2At + B) - 8(At^2 + Bt + C) = (-8A)t^2 + (-14A - 8B)t + (2A - 7B - 8C)$$

Comparing coefficients, we find that $-8A = -8$, $-14A - 8B = -6$, and $2A - 7B - 8C = 1$. This gives $A = 1$, $-8B = 14(1) - 6 = 8$ or $B = -1$, and $-8C = -2(1) + 7(-1) + 1 = -8$ or $C = 1$. Hence $y_p = t^2 - t + 1$ is a particular solution, and the general solution of the differential equation is $y = y_h + y_p = C_1 e^{8t} + C_2 e^{-t} + t^2 - t + 1$.

(b) The differential equation $4t^2 y^3 y' = 1$ is separable, and we separate the variables by dividing the equation with t^2 before we integrate both sides:

$$4y^3 \cdot y' = \frac{1}{t^2} \quad \Rightarrow \quad \int 4y^3 \, \mathrm{d}y = \int t^{-2} \, \mathrm{d}t \quad \Rightarrow \quad y^4 = -t^{-1} + C = -\frac{1}{t} + C$$

The initial condition $y(1) = 1$ gives $1^4 = -1/1 + C$, or $C = 2$. We solve for y to get $y = \pm\sqrt[4]{2 - 1/t}$, and notice that only the positive solution satisfies the initial condition $y(1) = 1$. We conclude that the solution is

$$y = \sqrt[4]{2 - \frac{1}{t}}$$

(c) The differential equation $2y - 3t^2 + 2(t+1)y' = 0$ can be written in the form $p + q \cdot y' = 0$, and we try to solve it as an exact differential equation by finding a function $h = h(t, y)$ such that

$$h'_t = 2y - 3t^2, \quad h'_y = 2(t+1) = 2t + 2$$

The first condition gives $h = 2yt - t^3 + C(y)$, and when we substitute this expression into the second condition, it becomes $h'_y = 2t + C'(y) = 2t + 2$. This means that $C'(y) = 2$, and we find the solution $C(y) = 2y$. Hence the differential equation is exact with general solution

$$h(t, y) = 2yt - t^3 + 2y = C \quad \Rightarrow \quad y(2t+2) = C + t^3 \quad \Rightarrow \quad y = \frac{C + t^3}{2t + 2}$$

(d) The matrix A has eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 4$ from Question 2(c). Since A is symmetric and therefore diagonalizable, we know that we have a base $\{\mathbf{v}_1, \mathbf{v}_2\}$ of E_1 and a base $\{\mathbf{v}_3\}$ of E_4. We compute these base vectors using Gaussian elimination:

$$E_1: \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_4: \begin{pmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 \\ -2 & 1 & -1 \\ -1 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

In E_1 we see that $x + y - z = 0$, or $x = -y + z$ with y, z free. We may therefore choose the base vectors $\mathbf{v}_1 = (-1, 1, 0)$ and $\mathbf{v}_2 = (1, 0, 1)$ of E_1. In E_4 we see that $-3y - 3z = 0$, or $y = -z$, and $x - 2y - z = 0$, or $x = 2(-z) + z = -z$, with z free. We may therefore choose the base vector $\mathbf{v}_3 = (-1, -1, 1)$ of E_4. This gives

$$\mathbf{y}_t = C_1 \mathbf{v}_1 \cdot 1^t + C_2 \mathbf{v}_2 \cdot 1^t + C_3 \mathbf{v}_3 \cdot 4^t = C_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \cdot 4^t$$

Question 5.

(a) We notice that objective function $f(x, y, z) = x^2 - xy + xz$ is a quadratic form with symmetric matrix A, and that the function $g(x, y, z) = y^2 - yz + z^2$ defining the constraint is a quadratic form with symmetric matrix D, where A and D are given by

$$A = \begin{pmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix}$$

We rewrite the Kuhn-Tucker problem to one in standard form, which can be stated as

$$\max -f(\mathbf{x}) = -\mathbf{x}^T A \mathbf{x} \text{ when } g(\mathbf{x}) = \mathbf{x}^T D \mathbf{x} \le 12$$

It has Lagrangian $\mathcal{L}(x, y, z; \lambda) = -\mathbf{x}^T A \mathbf{x} - \lambda(\mathbf{x}^T D \mathbf{x} - 12)$ when we write it in matrix form, and the first order conditions are $-2A\mathbf{x} - \lambda(2D\mathbf{x}) = \mathbf{0}$, or $2(A + \lambda D)\mathbf{x} = \mathbf{0}$ after multiplying the equation with -1. The constraint is $\mathbf{x}^T D \mathbf{x} \le 12$ in matrix form, and the CSC's are $\lambda \ge 0$ and $\lambda = 0$ if $\mathbf{x}^T D \mathbf{x} < 12$. Alternatively, you could write the first order conditions as $(A + \lambda D)\mathbf{x} = \mathbf{0}$ (dividing the equation by 2). Alternatively, one could specify the Kuhn-Tucker conditions without using matrices (see below).

(b) The first order conditions give a homogeneous 3×3 linear system with parameter λ, and we have that $|2(A + \lambda D)| = 0$ or $\mathbf{x} = \mathbf{0}$. If $\mathbf{x} = \mathbf{0}$ then $\mathbf{x}^T D \mathbf{x} = 0 < 12$ is non-binding, and $\lambda = 0$. This gives the candidate point $(0, 0, 0; 0)$ with $-f(0, 0, 0) = 0$. Otherwise, we have $|2(A + \lambda D)| = 0$, and to find candidate points in this case we solve the equation

$$|2(A + \lambda D)| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2\lambda & -\lambda \\ 1 & -\lambda & 2\lambda \end{vmatrix} = 0$$

Using cofactor expansion along the first row, this gives

$$\begin{aligned} |2(A + \lambda D)| &= 2(4\lambda^2 - \lambda^2) - (-1)(-2\lambda + \lambda) + 1(\lambda - 2\lambda) \\ &= 6\lambda^2 + (-\lambda) + (-\lambda) = 6\lambda^2 - 2\lambda = 2\lambda(3\lambda - 1) = 0 \end{aligned}$$

This means that $\lambda = 0$ or $\lambda = 1/3$. In case $\lambda = 0$, we have the first order conditions

$$2x - y + z = 0, \quad -x = 0, \quad x = 0$$

which gives $x = 0$ and $y = z$, and the constraint gives $z^2 - z^2 + z^2 = z^2 \leq 12$. This gives the candidate points $(0, z, z; 0)$ with $z^2 \leq 12$ and $-f(0, z, z) = 0$. Finally, in case $\lambda = 1/3$, then $y^2 - yz + z^2 = 12$ is binding by the CSC (since $\lambda > 0$), and we solve the first order conditions with $\lambda = 1/3$ using Gaussian elimination:

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2/3 & -1/3 \\ 1 & -1/3 & 2/3 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1/6 & 1/6 \\ 0 & 1/6 & 1/6 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence z is free, $y + z = 0$, or $y = -z$, and $2x - y + z = 0$, or $2x = (-z) - z = -2z$, which gives $x = -z$. The constraint then gives $(-z)^2 - (-z)z + z^2 = 3z^2 = 12$, or $z^2 = 4$. Hence $z = 2$ or $z = -2$, and we find the candidate points $(-2, -2, 2; 1/3)$ and $(2, 2, -2; 1/3)$ with $-f(2, 2, -2) = -(4 - 4 + (-4)) = 4$ and $-f(-2, -2, 2) = -(4 - 4 + (-4)) = 4$. The two candidate points with $\lambda = 1/3$ are therefore the best candidates for maximum of $-f$, and we use the SOC to test if they are maxima: The function $h(\mathbf{x}) = \mathcal{L}(\mathbf{x}; 1/3) = -\mathbf{x}^T A\mathbf{x} - 1/3 \cdot \mathbf{x}^T D\mathbf{X}$ has Hessian matrix

$$H(h) = -2A - 2/3 \cdot D = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -2/3 & 1/3 \\ -1 & 1/3 & -2/3 \end{pmatrix}$$

The leading principal minors of $H(h)$ are $D_1 = -2$, $D_2 = 4/3 - 1 = 1/3$ and $D_3 = 0$ (since the Gaussian process above shows that $-H(h)$ has rank two). By the RRC, it follows that $H(h)$ is negative semidefinite, and it follows that h is concave. By the SOC, this means that $-f_{\max} = 4$, or the $f_{\min} = -4$ in the original Kuhn-Tucker problem, with minimum points $(-2, -2, 2)$ and $(2, 2, -2)$ where the Lagrange multiplier $\lambda = 1/3$.

Alternatively, we could find candidate points without using matrices: We write the Lagrangian as $\mathcal{L}(x, y, z; \lambda) = -x^2 + xy - xz - \lambda(y^2 - yz + z^2 - 12)$, and the first order conditions are

$$\begin{aligned} \mathcal{L}'_x &= -2x + y - z = 0 \\ \mathcal{L}'_y &= x - \lambda(2y - z) = 0 \\ \mathcal{L}'_z &= -x - \lambda(-y + 2z) = 0 \end{aligned}$$

Adding the last two equations, we get $-\lambda(y + z) = 0$. If $\lambda > 0$, this gives $y = -z$. When we substitute $y = -z$ in the FOC's, we get $x = -z$ in the first equation, and $\lambda = 1/3$ from the last two equations (since $z = 0$ gives the point $(0, 0, 0)$ where the constraint is not binding, which contradicts $\lambda > 0$). Binding constraint gives $y^2 - yz + z^2 = (-z)^2 - (-z)z + z^2 = 3z^2 = 12$, or $z^2 = 4$ and $z = \pm 2$. We obtain the two candidate points with $\lambda > 0$:

$$(x, y, z; \lambda) = (-2, -2, 2; 1/3), \ (2, 2, -2; 1/3)$$

To show that these candidate points are maximum points in the Kuhn-Tucker problem in standard form, or minimum points in the original problem, we use the SOC in the same way as before (see above).