Exam	Final exam in GRA 6035 Mathematics
Date	November 25th 2024 at 0900 - 1400

You must give reasons for your answers.

Question 1.

(a) **(3p)** Find the general solution of the differential equation $y'' - 3y' - 10y = 0$.

(b) **(3p)** Determine the definiteness of the quadratic form $f(x, y, z) = x^2 + 4xy + 2xz + 3y^2 + 2yz$.

(c) **(3p)** Find the rank of the matrix A for all values of the parameter s:

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 2 \\ 1 & 3 & s & 5 \end{pmatrix}$$

(d) **(3p)** Is the set $D = \{(x, y, z) : x^2 + 2y^2 - 3z^2 \leq 6\}$ compact?

Question 2.

We consider the matrix A and the vector $\mathbf{v}$ given by

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 5 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

(a) **(6p)** Compute the trace and the determinant of A.

(b) **(6p)** Show that $\mathbf{v}$ is an eigenvector of A, and determine its eigenvalue.

(c) **(6p)** Find all the eigenvalues of A.

(d) **(6p)** Determine the dimension of the vector space W of all vectors in $\mathbb{R}^3$ orthogonal to $\mathbf{v}$, and find a base of W consisting of pairwise orthogonal vectors.

Question 3.

(a) **(6p)** Find the stable equilibrium states of the differential equation $y' = y(1 - y)$.

(b) **(6p)** Find the solution of the differential equation $y' = 2ty^2$ that satisfies $y(0) = 1$.

(c) **(6p)** Find the general solution of the system of differential equations:

$$y_1' = 8y_1 - 3y_2 + 2$$
$$y_2' = 2y_1 + y_2 + 4$$

Question 4.

Let $u(x, y, z) = x^2 + 2xy - 2xz + 2y^2 - 2yz + z^2 + 2x + 4y - 2z + 4$ and consider the function

$$f(x, y, z) = u \ln u - 2u \text{ with } u = u(x, y, z)$$

(a) **(6p)** Find the stationary points of u.

(b) **(6p)** Find the maximum and minimum value of u, if they exist.

(c) **(6p)** Determine the range of f.

Question 5.

Consider the Kuhn-Tucker problem

$$\min f(x, y, z) = x^2 + y^2 + z^2 \text{ when } xy - xz - yz \geq 4$$

(a) **(6p)** Write down the Kuhn-Tucker conditions for this problem.

(b) **(6p)** Find the minimum value in the Kuhn-Tucker problem, if it exists.