Solutions	Final exam in GRA 6035 Mathematics
Date	November 25th 2024 at 0900 - 1400

Question 1.

(a) The differential equation $y'' - 3y' - 10y = 0$ is a linear second order differential equation that is homogeneous, and it has characteristic equation $r^2 - 3r - 10 = 0$. The characteristic roots are $r_1 = 5$ and $r_2 = -2$, since $5 + (-2) = 3$ and $5(-2) = -10$. Hence the general solution is

$$y = C_1 \cdot e^{5t} + C_2 \cdot e^{-2t}$$

(b) The quadratic form $f(x, y, z) = x^2 + 4xy + 2xz + 3y^2 + 2yz$ has symmetric matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The leading principal minor $D_2 = 3 - 4 = -1 < 0$, hence the quadratic form is indefinite.

(c) We use Gaussian elimination to find an echelon form E of the matrix A:

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 2 \\ 1 & 3 & s & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & s-3 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & s-4 & 4 \end{pmatrix} = E$$

We see that the last row of E has a pivot position in the third column if $s \neq 4$, and in the fourth column if $s = 4$. Hence the rank of A is $\operatorname{rk} A = 3$ for all values of s.

(d) The set $D = \{(x, y, z) : x^2 + 2y^2 - 3z^2 \leq 6\}$ is closed, since it is given by a closed inequality. We see that when $x = y = 0$, the inequality defining D becomes $-3z^2 \leq 6$, which is satisfied for all values of z. Hence $(0, 0, z)$ is in D for all values of z, and this means that D is not bounded, and therefore not compact.

Question 2.

(a) The trace of A is $\operatorname{tr}(A) = 1 + 2 + 5 = 8$, and we find the determinant of A using cofactor expansion along the last row:

$$\det(A) = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 5 \end{vmatrix} = -2(0 + 4) + 5(2 - 4) = -8 - 10 = -18$$

This gives $\operatorname{tr}(A) = 8$ and $\det(A) = -18$.

(b) To show that $\mathbf{v}$ is an eigenvector of A, we compute $A\mathbf{v}$ and compare it with $\lambda\mathbf{v}$:

$$A\mathbf{v} = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}, \quad \lambda\mathbf{v} = \lambda \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\lambda \\ -2\lambda \\ \lambda \end{pmatrix}$$

We see that $A\mathbf{v} = \lambda\mathbf{v}$ when $\lambda = -1$, hence $\mathbf{v}$ is an eigenvector of A with eigenvalue $\lambda = -1$.

(c) Since A is symmetric, we know that A has three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (where eigenvalues are repeated according to their multiplicities), and from (b) when know that $\lambda_1 = -1$ is one of the eigenvalues. Using the equations

$$\lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr}(A) = 8, \quad \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det(A) = -18$$

we find that $\lambda_2 + \lambda_3 = 8 - (-1) = 9$ and $\lambda_2\lambda_3 = -18/(-1) = 18$. This gives $\lambda_2 = 3$ and $\lambda_3 = 6$, and we conclude that the eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 3$, and $\lambda_3 = 6$.

(d) We have that $\mathbf{w} = (x, y, z)$ is orthogonal to $\mathbf{v}$ if and only if $\mathbf{v} \cdot \mathbf{w} = 3x - 2y + z = 0$, hence $W = \operatorname{Null}(B)$ with $B = \begin{pmatrix} 3 & -2 & 1 \end{pmatrix}$. Since $\operatorname{rk} B = 1$, we have that $\dim W = 3 - 1 = 2$. We know that $\dim E_3 = \dim E_6 = 1$ since the eigenvalues $\lambda = 3$ and $\lambda = 6$ have multiplicity one. Therefore we can find a base of E_3 consisting of one vector $\mathbf{w}_1$, and a base of E_6 consisting of one vector $\mathbf{w}_2$. Moreover, since A is symmetric, and $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2$ are eigenvectors of A with different eigenvalues, we have that $\mathbf{v} \cdot \mathbf{w}_1 = \mathbf{v} \cdot \mathbf{w}_2 = \mathbf{w}_1 \cdot \mathbf{w}_2 = 0$. In other words, $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal vectors in W. We find these vectors explicitly using Gaussian elimination:

$$E_3 : \begin{pmatrix} -2 & 2 & -2 \\ 2 & -1 & 0 \\ -2 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} -2 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \qquad E_6 : \begin{pmatrix} -5 & 2 & -2 \\ 2 & -4 & 0 \\ -2 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} -2 & 0 & -1 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

This gives $\mathbf{w}_1 = (1, 2, 1)$ and $\mathbf{w}_2 = (2, 1, -4)$ (or non-zero scalar multiples of these vectors). It follows that

$$\mathcal{B} = \{\mathbf{w}_1, \mathbf{w}_2\} \text{ with } \mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \ \mathbf{w}_2 = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$$

is a base of the 2-dimensional vector space W consisting of pairwise orthogonal vectors.

Question 3.

(a) The differential equation is autonomous and can be written $y' = F(y)$ with $F(y) = y(1-y)$. Since $F(y) = y(1-y) = 0$ gives $y = 0$ or $y = 1$, these are the equilibrium states. To check whether they are stable, we use the Stability Theorem, and compute

$$F'(y) = (y(1-y))' = (y - y^2)' = 1 - 2y$$

This gives $F'(0) = 1 > 0$, or that $y_e = 0$ is unstable, and $F'(1) = -1 < 0$, or that $y_e = 1$ is stable. We conclude that the stable equilibrium states are $y_e = 1$.

(b) The differential equation $y' = 2ty^2$ is separable, and we separate the variables before we integrate both sides:

$$y' = 2ty^2 \quad \Rightarrow \quad \frac{1}{y^2}\, y' = 2t \quad \Rightarrow \quad \int \frac{1}{y^2}\, \mathrm{d}y = \int 2t\, \mathrm{d}t \quad \Rightarrow \quad -\frac{1}{y} = t^2 + C$$

The initial condition $y(0) = 1$ gives $-1/1 = 0^2 + C$, or $C = -1$. We write the solution in explicit form and obtain

$$-\frac{1}{y} = t^2 - 1 \quad \Rightarrow \quad \frac{1}{y} = 1 - t^2 \quad \Rightarrow \quad y = \frac{1}{1-t^2}$$

(c) We write the system of differential equations in the form $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$, where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A = \begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

To solve the homogeneous equation, we find the eigenvalues and eigenvectors of A. The characteristic equation is $\lambda^2 - 9\lambda + 14 = 0$ since A has trace 9 and determinant 14. We factorize this as $(\lambda - 2)(\lambda - 7) = 0$, which gives $\lambda_1 = 2$ and $\lambda_2 = 7$. Since each eigenvalue has multiplicity one, there is a base $\mathbf{v}_i$ for E_{λ_i} which we can find using Gaussian elimination:

$$E_2 : \begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} \to \begin{pmatrix} 6 & -3 \\ 0 & 0 \end{pmatrix} \qquad E_7 : \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

We may choose the base vectors $\mathbf{v}_1 = (1, 2)$ and $\mathbf{v}_2 = (3, 1)$. This gives

$$\mathbf{y}_h = C_1 \mathbf{v}_1 e^{\lambda_1 t} + C_2 \mathbf{v}_2 e^{\lambda_2 t} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot e^{2t} + C_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot e^{7t}$$

We find the equilibrium state by solving $A\mathbf{y} + \mathbf{b} = \mathbf{0}$, which is a linear system $A\mathbf{y} = -\mathbf{b}$ that we can solve using Gaussian elimination:

$$\left(\begin{array}{cc|c} 8 & -3 & -2 \\ 2 & 1 & -4 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 1 & -4 \\ 8 & -3 & -2 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 1 & -4 \\ 0 & -7 & 14 \end{array}\right)$$

Back substitution gives $-7y_2 = 14$, or $y_2 = -2$, and $2y_1 - 2 = -4$, or $y_1 = -1$. The equilibrium state is therefore $\mathbf{y}_e = (-1, -2)$, and since $\mathbf{y}_p = \mathbf{y}_e$ is a particular solution of the inhomogeneous system of differential equations $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$, the general solution is given by

$$\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot e^{2t} + C_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot e^{7t} + \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

Question 4.

(a) We write $u(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C$ in matrix form, with

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 4 & -2 \end{pmatrix}, \quad C = 4$$

The first order conditions for u are $2A\mathbf{x} + B^T = \mathbf{0}$, which can be written $2A\mathbf{x} = -B^T$. We solve this linear system using Gaussian elimination:

$$\left(\begin{array}{ccc|c} 2 & 2 & -2 & -2 \\ 2 & 4 & -2 & -4 \\ -2 & -2 & 2 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 2 & -2 & -2 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We see that z is free, $2y = -2$, or $y = -1$ and $2x + 2(-1) - 2z = -2$, or $x = z$. There are therefore infinitely many stationary points $(x, y, z) = (z, -1, z)$ with z free.

(b) Since the leading principal minors of A are $D_1 = 1$, $D_2 = 2 - 1 = 1$ and $D_3 = 0$, for example using the echelon form of $2A$ found in (a), we see that A is positive semi-definite by the RRC. This means that u is convex, and the stationary points found in (a) are minimum points. The minimum value of u is $u_{\min} = u(0, -1, 0) = 2$ since $z = 0$ yields one of the minimum points $(0, -1, 0)$. Clearly, u has no maximum value, since $y = z = 0$ gives $f(x, 0, 0) = x^2 + 2x + 4 \to \infty$ when $x \to \infty$. In other words, the range of u is $V_u = [2, \infty)$.

(c) We consider the outer function $f(u) = u\ln(u) - 2u$ as a function of one variable with domain of definition equal to the range of u, which can be written $u \geq 2$. We compute the derivative of the outer function:

$$f'(u) = 1\ln(u) + u(1/u) - 2 = \ln(u) - 1$$

Hence $f'(u) = 0$ when $\ln(u) - 1 = 0$, and this gives $\ln(u) = 1$, or $u = e^1 = e > 2$. We see that f is decreasing in the interval $[2, e]$ and increasing in $[e, \infty)$. This means that $u = e$ is a minimum point for f, and $f_{\min} = f(e) = e\ln(e) - 2e = e - 2e = -e$. To determine whether f has a maximum, we compute $f(2) = 2\ln 2 - 2 = 2(\ln 2 - 1)$ and notice that $f(u) = u(\ln u - 2) \to \infty$ when $u \to \infty$. We conclude that f has no maximum value, and its range is $V_f = [-e, \infty)$.

Question 5.

(a) We notice that objective function is a quadratic form with symmetric matrix I, and that the function $g(x, y, z) = xy - xz - yz$ defining the constraint is a quadratic form with symmetric matrix

$$A = \begin{pmatrix} 0 & 1/2 & -1/2 \\ 1/2 & 0 & -1/2 \\ -1/2 & -1/2 & 0 \end{pmatrix}$$

We rewrite the Kuhn-Tucker problem to one in standard form, which can be stated as

$$\max -f(x, y, z) = -x^2 - y^2 - z^2 \text{ when } -g(x, y, z) = -xy + xz + yz \leq -4$$

It has Lagrangian $\mathcal{L}(x, y, z; \lambda) = -\mathbf{x}^T I\mathbf{x} - \lambda(-\mathbf{x}^T A\mathbf{x} + 4)$ when we write it in matrix form, and the first order conditions are $-2I\mathbf{x} - \lambda(-2A\mathbf{x}) = \mathbf{0}$, or $2(I - \lambda A)\mathbf{x} = \mathbf{0}$ after multiplying the equation with -1. The constraint is $\mathbf{x}^T A\mathbf{x} \geq 4$ in matrix form, and the CSC's are $\lambda \geq 0$ and $\lambda = 0$ if $\mathbf{x}^T A\mathbf{x} > 4$. Alternatively, you could write the first order conditions as $(I - \lambda A)\mathbf{x} = \mathbf{0}$ (dividing the equation by 2), the constraint as $-\mathbf{x}^T A\mathbf{x} \leq -4$ and the CSC's as $\lambda \geq 0$ and $\lambda = 0$ if $-\mathbf{x}^T A\mathbf{x} < -4$ (using the constraint from the standard form rather than the equivalent original form).

(b) The first order conditions give a homogeneous 3×3 linear system with parameter λ, and we have that $|I - \lambda A| = 0$ or $\mathbf{x} = 0$. Since $\mathbf{x} = \mathbf{0}$ does not satisfy the constraint, we solve the equation

$$|I - \lambda A| = \begin{vmatrix} 1 & -\lambda/2 & \lambda/2 \\ -\lambda/2 & 1 & \lambda/2 \\ \lambda/2 & \lambda/2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -t & t \\ -t & 1 & t \\ t & t & 1 \end{vmatrix} = 0$$

where we write $t = \lambda/2$ for simplicity. Using cofactor expansion along the first row, this gives

$$\begin{aligned}|I - \lambda A| &= 1(1-t^2) + t(-t-t^2) + t(-t^2-t)\\ &= (1-t)(1+t) + t(-t)(1+t) + t(-t)(t+1) = (t+1)(1-t-2t^2)\\ &= (t+1)(1+t)(1-2t) = -2(t+1)^2(t-1/2) = 0\end{aligned}$$

This means that $t = -1$ or $t = 1/2$, or that $\lambda = -2$ or $\lambda = 1$. By the CSC, we have that $\lambda \geq 0$, and therefore we must have $\lambda = 1$. We solve the first order conditions for $\lambda = 1$ using Gaussian elimination:

$$I - A = \begin{pmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1/2 & 1/2 \\ 0 & 3/4 & 3/4 \\ 0 & 3/4 & 3/4 \end{pmatrix} \to \begin{pmatrix} 1 & -1/2 & 1/2 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence z is free, $3y+3z = 0$, or $y = -z$, and $x - 1/2(-z) + 1/2(z) = 0$, or $x = -z$. This means that the points $(-z, -z, z; 1)$ satisfy the first order conditions. The CSC's give $xy - xz - yz = 4$ since $\lambda > 0$, or $(-z)^2 - (-z)z - (-z)z = 3z^2 = 4$, or $z = \pm 2/\sqrt{3}$. We conclude that there are two candidate points that satisfies the Kuhn-Tucker conditions:

$$(x, y, z; \lambda) = (-2/\sqrt{3}, -2/\sqrt{3}, 2/\sqrt{3}; 1),\ (2/\sqrt{3}, 2/\sqrt{3}, -2/\sqrt{3}; 1)$$

These candidate points have $-f = -4/3 \cdot 3 = -4$. We test whether these are maximum points using the SOC, and let $h(x, y, z) = \mathcal{L}(x, y, z; 1)$. This gives

$$h(x, y, z) = -\mathbf{x}^T I \mathbf{x} - 1(-\mathbf{x}^T A \mathbf{x} + 4) \quad \Rightarrow \quad H(h) = -2I + 2A = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix}$$

The leading principal minors are $D_1 = -2$, $D_2 = 3$, and since $|I - A| = 0$ from the Gaussian elimination above, $D_3 = |-2(I - A)| = (-2)^3 |I - A| = 0$. It follows by the RRC that $H(h)$ is negative semidefinite and that h is concave, and from the SOC that the two candidate points are maximum points in the Kuhn-Tucker problem in standard form. This means that $-f_{\max} = -4$ is the maximum value of $-f$, and therefore $f_{\min} = 4$ is the minimum value in the original Kuhn-Tucker problem. As an alternative method, rather solving than $|I - \lambda A| = 0$ to find λ (as we have done above), we could solve the equation

$$|2(I - \lambda A)| = \begin{vmatrix} 2 & -\lambda & \lambda \\ -\lambda & 2 & \lambda \\ \lambda & \lambda & 2 \end{vmatrix} = 0$$

With this approach, we would find the factorization of the left-hand side as

$$\begin{aligned}&2(4-\lambda^2) + \lambda(-2\lambda - \lambda^2) + \lambda(-\lambda^2 - 2\lambda) = 2(2-\lambda)(2+\lambda) - 2\lambda^2(\lambda+2)\\ &= 2(\lambda+2)(2-\lambda-\lambda^2) = -2(\lambda+2)(\lambda^2+\lambda-2) = -2(\lambda+2)(\lambda+2)(\lambda-1)\end{aligned}$$

Hence we would obtain the Lagrange multipliers $\lambda = -2$ and $\lambda = 1$ directly.