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| Solutions | Final exam in GRA 6035 Mathematics |
| Date | January 10th 2024 at 0900 - 1400 |

**Question 1.**

(a) The second order differential equation $y'' - 2y' = 0$ is homogeneous, and it has characteristic equation $r^2 - 2r = r(r-2) = 0$ with characteristic roots $r_1 = 0$ and $r_2 = 2$. The general solution is

$$y = C_1 \cdot e^{0 \cdot t} + C_2 \cdot e^{2t} = C_1 + C_2 \cdot e^{2t}$$

(b) We form the matrix $A = (\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)$, and find the pivot positions:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 8 & 5 \\ 1 & 3 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there are pivots positions in all three columns, the three vectors are linearly independent.

(c) The first order derivatives of $f$ and the first order conditions are given by

$$f'_x = -3x^2 + 3y + 3z = 0, \quad f'_y = 3x - 3y^2 + 3z = 0, \quad f'_z = 3x + 3y - 3z^2 = 0$$

Since $3(-2^2 + 2 + 2) = 0$, we see that $(2, 2, 2)$ is a stationary point of $f$. The Hessian matrix of $f$ at this point is given by

$$H(f) = \begin{pmatrix} -6x & 3 & 3 \\ 3 & -6y & 3 \\ 3 & 3 & -6z \end{pmatrix} \quad \Rightarrow \quad H(f)(2,2,2) = \begin{pmatrix} -12 & 3 & 3 \\ 3 & -12 & 3 \\ 3 & 3 & -12 \end{pmatrix}$$

Since $D_1 = -12$, $D_2 = 144 - 9 = 135$, and $D_3 = 3(9 + 36) - 3(-36 - 9) - 12(135) = -1350$, it follows that $H(f)(2, 2, 2)$ is negative definite. By the second derivative test, it follows that $(2, 2, 2)$ is a local maximum point of $f$.

(d) Since $A$ is a symmetric matrix with $\mathrm{tr}(A) = 4$ and $\det(A) = 1(-4-4)-1(4-2)+1(2+1) = -7$, it has three eigenvalues with sum equal to 4 and product equal to $-7$. The equilibrium states are given by $A\mathbf{y} + \mathbf{b} = \mathbf{0}$, or $A\mathbf{y} = -\mathbf{b}$. Since $|A| \neq 0$, there is a unique stable equilibrium state $\mathbf{y}_e = A^{-1}(-\mathbf{b})$. We know that it is stable if and only if $A$ has three negative eigenvalues. This is not the case since the sum of the eigenvalues is $\mathrm{tr}(A) = 4$. The system of differential equations therefore has no stable equilibrium state.

**Question 2.**

(a) We compute the determinant of $A$ using cofactor expansion along the last row:

$$|A| = \begin{vmatrix} 2 & 7 & 3 \\ 3 & 11 & 5 \\ 1 & -4 & 0 \end{vmatrix} = 1(35 - 33) + 4(10 - 9) = 2 + 4 = 6$$

Since $A$ is a $3 \times 3$ matrix with $|A| \neq 0$, we have that $\mathrm{rk}(A) = 3$.

(b) We can write the equation $A\mathbf{x} = \mathbf{x}$ as $A\mathbf{x} - \mathbf{x} = A\mathbf{x} - I\mathbf{x} = (A - I)\mathbf{x} = \mathbf{0}$. To solve this homogeneous linear system, we use Gaussian elimination to find an echelon form of $A - I$:

$$\begin{pmatrix} 2-1 & 7 & 3 \\ 3 & 11-1 & 5 \\ 1 & -4 & 0-1 \end{pmatrix} = \begin{pmatrix} 1 & 7 & 3 \\ 3 & 10 & 5 \\ 1 & -4 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 7 & 3 \\ 0 & -11 & -4 \\ 0 & -11 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 7 & 3 \\ 0 & -11 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that $z$ is a free variable, and back substitution gives that $-11y - 4z = 0$, or $y = -4z/11$, and that $x + 7y + 3z = x + 7(-4z/11) + 3z = 0$, or $x = 28z/11 - 33z/11 = -5z/11$. The solutions of the linear system can therefore be written

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5z/11 \\ -4z/11 \\ z \end{pmatrix} = \frac{z}{11} \cdot \begin{pmatrix} -5 \\ -4 \\ 11 \end{pmatrix} = \frac{z}{11} \cdot \mathbf{w} \text{ with } \mathbf{w} = \begin{pmatrix} -5 \\ -4 \\ 11 \end{pmatrix}$$

There are infinitely many solutions (one degree of freedom) since the set of solutions can be written as $\mathrm{span}(\mathbf{w})$ with $\mathbf{w} = (-5, -4, 11)$.

(c) The characteristic equation of $A$ is $-\lambda^3+\mathrm{tr}(A)\cdot\lambda^2-c_2\cdot\lambda+\det(A) = -\lambda^3+13\lambda^2-18\lambda+6=0$, since $\mathrm{tr}(A)=2+11+0=13$, $c_2=M_{12}+M_{23}+M_{13}=1+20-3=18$, and $\det(A)=6$ from (a). We know that $\lambda=1$ is a solution since $A\mathbf{x}=1\cdot\mathbf{x}$ has non-trivial solutions from (b). Alternatively, we can see this directly by substituting $\lambda=1$ into the characteristic equation. This gives that

$$-\lambda^3+13\lambda^2-18\lambda+6=(\lambda-1)\cdot p(\lambda)=(\lambda-1)(-\lambda^2+12\lambda-6)=0$$

where we have found the second factor $p(\lambda)$ using polynomial division:

$$(-\lambda^3+13\lambda^2-18\lambda+6):(\lambda-1)=-\lambda^2+12\lambda-6$$

The equation $-\lambda^2+12\lambda-6=0$ can be written as $\lambda^2-12\lambda+6=0$, and we solve it using the quadratic formula:

$$\lambda=\frac{12\pm\sqrt{12^2-4(6)}}{2}=6\pm\frac{1}{2}\sqrt{120}=6\pm\sqrt{30}$$

Since $A$ has three distinct eigenvalues $\lambda=1$ and $\lambda=6\pm\sqrt{30}$, it follows that $A$ is diagonalizable.

**Question 3.**

(a) The difference equation $y_{t+2}+y_{t+1}-6y_t=3-4t$ is second order linear and can be solved using the superposition principle. To find the homogeneous solution $y_t^h$, we consider the characteristic equation $r^2+r-6=0$. Using the quadratic formula, we find that it has roots $r=2$ and $r=-3$, and $y_t^h=C_1\cdot 2^t+C_2\cdot(-3)^t$. To find a particular solution, we consider $y_t=At+B$, which gives $y_{t+1}=A(t+1)+B=At+A+B$ and $y_{t+2}=A(t+2)+B=At+2A+B$. When we substitute this into the difference equation, we get

$$(At+2A+B)+(At+A+B)-6(At+B)=3-4t\ \Rightarrow\ (-4A)t+(3A-4B)=-4t+3$$

Comparing coefficients, we find that $A=1$ and $3-4B=3$, or $B=0$. This gives $y_t^p=t$, and the general solution is

$$y_t=C_1\cdot 2^t+C_2\cdot(-3)^t+t$$

(b) The differential equation $t+y'=y$ can be written $y'-y=-t$, and it is therefore linear. It can be solved using the superposition principle since $a(t)=-1$ is a constant: Since the characteristic equation $r-1=0$ has root $r=1$, the homogeneous solution is $y_h=C\cdot e^t$. To find a particular solution, we consider $y=At+B$, which gives $y'=A$. When we substitute this into the differential equation, we get

$$A-(At+B)=-t\ \Rightarrow\ (-A)t+(A-B)=-t$$

Comparing coefficients, we find that $A=1$ and $1-B=0$, or $B=1$. This gives $y_p=t+1$, and the general solution is

$$y=Ce^t+t+1$$

Alternatively, we could have used integrating factor to solve the differential equation.

(c) The characteristic equation of $A$ is $\det(A-\lambda I)=0$, which gives

$$\begin{vmatrix}2-\lambda & 0\\ 1 & -1-\lambda\end{vmatrix}=(2-\lambda)(-1-\lambda)=0$$

and the two eigenvalues of $A$ are therefore $\lambda_1=2$ and $\lambda_2=-1$. Since each eigenvalue has multiplicity one, there is a base $\mathbf{v}_i$ for $E_{\lambda_i}$ which we can find using Gaussian elimination:

$$E_2:\begin{pmatrix}0 & 0\\ 1 & -3\end{pmatrix}\to\begin{pmatrix}1 & -3\\ 0 & 0\end{pmatrix}\qquad E_{-1}:\begin{pmatrix}3 & 0\\ 1 & 0\end{pmatrix}\to\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

We may choose the base vectors $\mathbf{v}_1=(3,1)$ and $\mathbf{v}_2=(0,1)$. We find the equilibrium state by solving $A\mathbf{y}_t+\mathbf{b}=\mathbf{y}_t$, since the equilibrium states are the constant solutions, with $\mathbf{y}_t=\mathbf{y}_{t+1}$. This gives $A\mathbf{y}_e-\mathbf{y}_e=-\mathbf{b}$, or $(A-I)\mathbf{y}_e=-\mathbf{b}$. We solve this linear system using Gaussian elimination:

$$\left(\begin{matrix}2-1 & 0\\ 1 & -1-1\end{matrix}\left|\begin{matrix}2\\ -1\end{matrix}\right.\right)=\left(\begin{matrix}1 & 0\\ 1 & -2\end{matrix}\left|\begin{matrix}2\\ -1\end{matrix}\right.\right)\to\left(\begin{matrix}1 & 0\\ 0 & -2\end{matrix}\left|\begin{matrix}2\\ -3\end{matrix}\right.\right)$$

Back substitution gives $-2y = -3$, or $y = 3/2$, and $x = 2$. The equilibrium state is therefore $(2, 3/2)$, and the general solution of the system of linear differential equations is

$$\mathbf{y}_t = \begin{pmatrix} 2 \\ 3/2 \end{pmatrix} + C_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot 2^t + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (-1)^t$$

(d) The differential equation $(t - 3y) + (8y - 3t) \cdot y' = 0$ is exact if there is a function $h = h(t, y)$ such that

$$h'_t = t - 3y, \quad h'_y = 8y - 3t$$

We see that $h = t^2/2 - 3yt + C(y)$ is the general solution of the first condition, and when we substitute this into the second condition, we find that $h'_y = -3t + C'(y)$, and therefore $h'_y = 8y - 3t$ when $C'(y) = 8y$. We choose the simplest solution is $C(y) = 4y^2$, which gives $h(t, y) = t^2/2 - 3yt + 4y^2$. Since the differential equation is exact of the form $h'_t + h'_y \cdot y' = 0$, the general solution is given by

$$h(t, y) = \frac{1}{2}t^2 - 3yt + 4y^2 = C \quad \Rightarrow \quad t^2 - 6yt + 8y^2 = 2C = K$$

The initial condition $y(1) = 0$ gives $(1)^2 - 6(0)(1) + 8(0)^2 = K$, or $K = 1$. Hence the particular solution in implicit form is given by

$$t^2 - 6yt + 8y^2 = 1 \quad \Rightarrow \quad 8y^2 - 6t \cdot y + (t^2 - 1) = 0$$

We solve the last equation using the quadratic formula, which gives that

$$y = \frac{6t \pm \sqrt{36t^2 - 4(8)(t^2 - 1)}}{2 \cdot 8} = \frac{6t \pm \sqrt{4t^2 + 32}}{16} = \frac{3t \pm \sqrt{t^2 + 8}}{8}$$

We see that the two solutions give $y(1) = 6/8$ or $y(1) = 0$, and therefore the particular solution that satisfies $y(1) = 0$ is given by

$$y = \frac{1}{8}\left(3t - \sqrt{t^2 + 8}\right)$$

**Question 4.**

(a) We write $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ and $g(\mathbf{x}) = \mathbf{x}^T D\mathbf{x}$, where $A$ and $D$ are the symmetric matrices of the objective function $f$ and the function $g(x, y, z, w) = x^2 + 2y^2 + 2z^2 + 6w^2$ that defines the constraint, with

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

The Lagrangian function of the problem is $\mathcal{L} = \mathbf{x}^T A\mathbf{x} - \lambda(\mathbf{x}^T D\mathbf{x} - 48)$, and the first order conditions are given by

$$\mathcal{L}'(\mathbf{x}) = 2A\mathbf{x} - \lambda(2D\mathbf{x}) = \mathbf{0} \quad \Rightarrow \quad A\mathbf{x} - \lambda D\mathbf{x} = (A - \lambda D)\mathbf{x} = \mathbf{0}$$

We find the solutions of the FOC's with $\lambda = 1$ by solving the linear system $(A - D)\mathbf{x} = \mathbf{0}$ using Gaussian elimination:

$$A - D = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & -6 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -5 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 1 & 1 & -5 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 2 & -6 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that $w$ is free, $-z + 3w = 0$, or $z = 3w$, $y - z + w = 0$, or $y = 2w$, and $-x + z + w = 0$, or $x = 4w$. The FOC's therefore give that $(x, y, z, w; \lambda) = (4w, 2w, 3w, w; 1)$, and the constraint

$x^2 + 2y^2 + 2z^2 + 6w^2 = 48$ gives $16w^2 + 8w^2 + 18w^2 + 6w^2 = 48$, or $48w^2 = 48$. It follows that $w^2 = 1$, or $w = \pm 1$, and we get two candidate points in the Lagrange problem with $\lambda = 1$:

$$(x, y, z, w; \lambda) = (4, 2, 3, 1; 1),\ (-4, -2, -3, -1; 1)$$

(b) The function $f$ is quadratic and can be written $f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$, where

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Its leading principal minors are $D_1 = D_2 = D_3 = D_4 = 0$. Using principal minors, we find that $A$ is indefinite since one of the principal 2-minors is negative:

$$\Delta_2 = \Delta_2^{23,23} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

(c) We test the candidate points in (a) using the Second Order Condition (SOC): We consider $h(\mathbf{x}) = \mathcal{L}(\mathbf{x}; 1) = \mathbf{x}^T (A - D)\mathbf{x} + 48\lambda$. Its Hessian is $H(h) = 2(A - D)$, where $A - D$ is the coefficient matrix of the linear system in (a). We notice that $|A - D| = 0$ since we found a free variable in (a). Moreover, the symmetric matrix

$$A - D = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & -6 \end{pmatrix}$$

had leading principal minors $D_1 = -1$, $D_2 = 2$, $D_3 = -1(4 - 1) + 1(0 + 2) = -1$, and $D_4 = |A - D| = 0$. Hence $A - D$ is negative semidefinite by the Reduced Rank Criterion (RRC), and $h$ is therefore a concave function. It follows from the SOC that the maximal value is

$$f_{\max} = f(4, 2, 3, 1) = f(-4, -2, -3, -1) = 2(12 + 4 + 6 + 2) = 48$$

and that $\mathbf{x} = (4, 2, 3, 1)\ (-4, -2, -3, -1)$ are maximum points in the Lagrange problem.

(d) We consider the Lagrange problem with parameter $a$ (where the case $a = 2$ is solved above):

$$\max f(x, y, z, w) = 2xz + 2xw + 2yz + 2yw \text{ when } x^2 + 2y^2 + az^2 + 6w^2 = 48$$

From (c) we know that $f^*(2) = 48$, since the maximal value is $f_{\max} = 48$ when $a = 2$. We compute the marginal change in the maximal value $f^*(a)$ using the Envelope Theorem: The Lagrangian of the problem with parameter $a$ is $\mathcal{L} = f(x, y, z, w) - \lambda(x^2 + 2y^2 + az^2 + 6w^2 - 48)$, and it follows that $\mathcal{L}'_a = -\lambda z^2$. Hence the marginal change at $a = 2$ is given by

$$\frac{\mathrm{d}f^*(a)}{\mathrm{d}a} = \mathcal{L}'_a(\mathbf{x}^*(a); \lambda^*(a)) = -\lambda^*(2) \cdot y^*(2)^2 = (-1) \cdot (\pm 3)^2 = -9$$

This gives the following estimate of maximal value when $a = 1$:

$$f^*(1) \approx f^*(2) + \Delta a \cdot \frac{\mathrm{d}f^*(a)}{\mathrm{d}a} = 48 + (-1) \cdot (-9) = 57$$

(e) We consider the FOC for any value of $\lambda$, given by $(A - \lambda D)\mathbf{x} = \mathbf{0}$. Since $\mathbf{x} = \mathbf{0}$ does not fit into the constraint, we must have $|A - \lambda D| = 0$. By dividing the last rows with suitable constants, we get

$$|A - \lambda D| = \begin{vmatrix} -\lambda & 0 & 1 & 1 \\ 0 & -2\lambda & 1 & 1 \\ 1 & 1 & -2\lambda & 0 \\ 1 & 1 & 0 & -6\lambda \end{vmatrix} = 0 \iff \begin{vmatrix} -\lambda & 0 & 1 & 1 \\ 0 & -\lambda & 1/2 & 1/2 \\ 1/2 & 1/2 & -\lambda & 0 \\ 1/6 & 1/6 & 0 & -\lambda \end{vmatrix} = \frac{0}{2 \cdot 2 \cdot 6} = 0$$

where the last equation is the characteristic equation of a new matrix $B$ with $\operatorname{tr}(B) = 0$ and $\operatorname{rk}(B) = 2$. Hence $\lambda = 0$ is an eigenvalue of $B$ of multiplicity $4 - 2 = 2$, and $\lambda = 1$ is an eigenvalue of $B$ by (a). The last eigenvalue $\lambda_4$ is given by $1 + 0 + 0 + \lambda_4 = 0$, or $\lambda_4 = -1$. The rest is very similar to (a) and (c) with $\lambda = -1$ instead of $\lambda = 1$: Going back to the first order conditions $(A + D)\mathbf{x} = \mathbf{0}$ for $\lambda = -1$, we get

$$A + D = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & -1 & 5 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -2 & 6 \end{pmatrix} \to \begin{pmatrix} \color{red}{1} & 0 & 1 & 1 \\ 0 & \color{red}{1} & 1 & -1 \\ 0 & 0 & \color{red}{-1} & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence $w$ is free, and back substitution gives $z = 3w$, $y = -2w$, and $x = -4w$. When we substitute these values into the constraint, we get $48w^2 = 48$, which gives $w^2 = 1$, or $w = \pm 1$. The candidate points with $\lambda = -1$ are therefore given by

$$(x, y, z, w; \lambda) = (-4, -2, 3, 1; -1), (4, 2, -3, -1; -1)$$

and $f(-4, -2, 3, 1) = f(4, 2, -3, -1) = 2(-12 - 4 - 6 - 2) = -48$. We use the SOC to check that these are minimum points: We have that $h(\mathbf{x}) = \mathcal{L}(\mathbf{x}; -1)$ has Hessian $H(h) = 2(A + D)$. Moreover,

$$A + D = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 6 \end{pmatrix}$$

has leading principal minors $D_1 = 1$, $D_2 = 2$, $D_3 = 3 - 2 = 1$, and $D_4|A + D| = 0$ since the linear system $(A + D)\mathbf{x} = \mathbf{0}$ has a free variable. This means that $A + D$ is positive semidefinite by the RRC, hence $h$ is convex. By the SOC, it follows that the minimum value in the Lagrange problem is

$$f_{\min} = f(-4, -2, 3, 1) = f(4, 2, -3, -1) = \color{blue}{-48}$$

Alternatively, we can argue that there is a minimum since the set of admissible points is compact, but it would be difficult to find the minimum value without finding the candidate points with $\lambda = -1$.