Solutions	Final exam in GRA 6035 Mathematics
Date	November 28th 2023 at 0900 - 1400

Question 1.

(a) The difference equation $y_{t+2} - y_{t+1} - 2y_t = 0$ is a linear difference equation that is homogeneous, and it has characteristic equation $r^2 - r - 2 = 0$. The characteristic roots are $r_1 = 2$ and $r_2 = -1$, hence the general solution is

$$y_t = C_1 \cdot 2^t + C_2 \cdot (-1)^t$$

(b) Since $A > 0$, the Markov chain is regular, and we compute the eigenvectors of A with eigenvalue $\lambda = 1$:

$$A - I = \begin{pmatrix} -0.06 & 0.14 \\ 0.06 & -0.14 \end{pmatrix} \to \begin{pmatrix} -0.06 & 0.14 \\ 0 & 0 \end{pmatrix}$$

Hence y is free, and $-0.06x + 0.14y = 0$, or $x = 0.14y/0.06 = 14y/6 = 7y/3$, and the eigenvectors in E_1 are given by $\mathbf{w} = (x, y) = (7y/3, y) = y/3 \cdot (7, 3)$. The equilibrium state of the Markov chain is the unique eigenvector in E_1 that is a state vector, and since $7 + 3 = 10$, it is given by

$$\mathbf{v} = \frac{1}{10} \cdot \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 7/10 \\ 3/10 \end{pmatrix}$$

(c) We consider the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{v}_4$, and use Gaussian elimination to find out how many solutions there are:

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 3 & 7 & -4 \\ 3 & 4 & 10 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & -6 \\ 0 & 1 & 1 & -6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since there is one degree of freedom and infinitely many solutions, there are infinitely many ways to write $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(d) We can write the function $f(x, y, z) = x^2 + 2y^2 + 5z^2 - 4xz + 2x - 6z + 5$ in matrix form as $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + B\mathbf{x} + 5$, with

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & -6 \end{pmatrix}$$

We see that A is positive definite since $D_1 = 1$, $D_2 = 2$, and $D_3 = |A| = 2(5 - 4) = 2$. This means that f is convex and any stationary point is a minimum point. We find the stationary points using $f'(\mathbf{x}) = 2A\mathbf{x} + B^T = \mathbf{0}$, or $A\mathbf{x} = -1/2 \cdot B^T$, and solve for $\mathbf{x}$ using Gaussian elimination:

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 2 & 0 & 0 \\ -2 & 0 & 5 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Back substitution gives $z = 1$, $2y = 0$, or $y = 0$, and $x - 2(1) = -1$, or $x = 1$. It follows that the minimum point of f is $(1, 0, 1)$, and the minimum value is

$$f_{\min} = f(1, 0, 1) = 3$$

Question 2.

(a) We use Gaussian elimination to find the rank of A:

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 1 & 5 \\ 1 & 18 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 2 \\ 0 & -7 & 1 \\ 0 & 14 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 2 \\ 0 & -7 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there are two pivot positions, we have that $\text{rk}(A) = 2$. Since there is no pivot in the third column, the determinant $\det(A) = 0$.

(b) To find the null space of A, we solve the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, and use the echelon form of A found above. We see that z is free, and back substitution gives $-7y+z=0$, or $y = z/7$, and $x + 4(z/7) + 2z = 0$, or $x = -4z/7 - 2z = -18z/7$. Hence the solutions are given by

$$\mathbf{x} = \begin{pmatrix} -18z/7 \\ z/7 \\ z \end{pmatrix} = \frac{z}{7} \cdot \begin{pmatrix} -18 \\ 1 \\ 7 \end{pmatrix}$$

It follows that $\mathbf{w} = (-18, 1, 7)$ is a base of Null(A).

(c) The characteristic equation of A is $-\lambda^3 + \text{tr}(A)\cdot\lambda^2 - c_2\cdot\lambda + \det(A) = -\lambda^3 + 2\lambda^2 + 99\lambda = 0$, since $\text{tr}(A) = 1+1+0 = 2$, $c_2 = M_{12} + M_{23} + M_{13} = -7 - 90 - 2 = -99$, and $\det(A) = 0$ from (a). This gives that

$$-\lambda^3 + 2\lambda^2 + 99\lambda = -\lambda(\lambda^2 - 2\lambda - 99) = -\lambda(\lambda - 11)(\lambda + 9) = 0$$

Therefore, the eigenvalue of A are $\lambda_1 = 0$, $\lambda_2 = 11$, and $\lambda_3 = -9$.

(d) If $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then we have

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}$$

Hence $\mathbf{v}$ is also an eigenvector of A^2 with eigenvalue λ^2. This proves that the eigenvalues of $B = A^2$ are $\lambda_1^2 = 0^2 = 0$, $\lambda_2^2 = 11^2 = 121$, and $\lambda_3^2 = (-9)^2 = 81$. Since all three eigenvalues of B are distinct and therefore of multiplicity one, it follows that $\dim \text{Null}(B) = 1$.

Question 3.

(a) The function f is quadratic and can be written $f(\mathbf{x}) = 27 + \mathbf{x}^T A\mathbf{x}$, where A is the symmetric matrix

$$A = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 \end{pmatrix}$$

Its leading principal minors are $D_1 = -1$, $D_2 = 2$, and $D_3 = -2(2-1) = -2$ and $D_4 = |A|$ is given by cofactor expansion along the last row:

$$|A| = 1 \cdot \begin{vmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 0 \end{vmatrix} + (-6)D_3 = 1(-1)(2-1) - 6(-2) = 11$$

It follows that A is negative definite, and therefore f is a concave function.

(b) We write the constraint as $\mathbf{x}^T D\mathbf{x} = 10$, since the function in the constraint is also a quadratic form (we use D for its symmetric matrix since we have used A for the symmetric matrix in the function f), where

$$D = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}$$

The Lagrangian function of the problem is $\mathcal{L} = 27 + \mathbf{x}^T A\mathbf{x} - \lambda(\mathbf{x}^T D\mathbf{x} - 10)$, and the first order conditions are given by

$$\mathcal{L}'(\mathbf{x}) = 2A\mathbf{x} - \lambda(2D\mathbf{x}) = \mathbf{0} \quad \Rightarrow \quad A\mathbf{x} - \lambda D\mathbf{x} = (A - \lambda D)\mathbf{x} = \mathbf{0}$$

We find the solutions of the FOC's with $\lambda = -2$ by solving the linear system $(A + 2D)\mathbf{x} = \mathbf{0}$ using Gaussian elimination:

$$\begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & -6 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -5 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 1 & 1 & -5 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 2 & -6 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that w is free, $-z+3w=0$, or $z=3w$, $y-z+w=0$, or $y=2w$, and $-x+z+w=0$, or $x=4w$. The FOC's therefore give that $(x,y,z,w;\lambda)=(4w,2w,3w,w;-2)$, and the constraint $xw+yz=10$ gives $(4w)w+(2w)(3w)=10$, or $10w^2=10$. It follows that $w^2=1$, or $w=\pm 1$, and we get two candidate points in the Lagrange problem with $\lambda=-2$:

$$(x,y,z,w;\lambda)=(4,2,3,1;-2),\ (-4,-2,-3,-1;-2)$$

(c) We test the candidate points in (b) using the Second Order Condition (SOC): We consider $h(\mathbf{x})=\mathcal{L}(\mathbf{x};-2)=27+\mathbf{x}^T(A+2D)\mathbf{x}$. Its Hessian is $H(h)=2(A+2D)$, where $A+2D$ is the coefficient matrix of the linear system in (b). We notice that $|A+2D|=0$ since we found a free variable in (b). Moreover, the symmetric matrix

$$A+2D=\begin{pmatrix}-1 & 0 & 1 & 1\\ 0 & -2 & 1 & 1\\ 1 & 1 & -2 & 0\\ 1 & 1 & 0 & -6\end{pmatrix}$$

had leading principal minors $D_1=-1$, $D_2=2$, $D_3=-1(4-1)+1(0+2)=-1$, and $D_4=|A+2D|=0$. Hence $A+2D$ is negative semidefinite by the Reduced Rank Criterion (RRC), and h is therefore a concave function. It follows from the SOC that

$$f_{\max}=f(4,2,3,1)=f(-4,-2,-3,-1)=27-16-8-18-6+24+4=7$$

is the maximal value and that $\mathbf{x}=(4,2,3,1)\ (-4,-2,-3,-1)$ are maximum points in the Lagrange problem.

(d) The set $D=\{(x,y,z,w):xw+yz=10\}$ is not compact since it is closed but not bounded. For example, we can see this by considering the points $(x,1,10,0)$: These points are in D for any value of x since $x\cdot 0+1\cdot 10=10$. This means that there is not a smallest or largest value of x among the points in D.

Question 4.

(a) The differential equation $y'+4ty=8t$ is linear and can be solved using the integrating factor $u=e^{2t^2}$ since $\int 4t\,\mathrm{d}t=2t^2+C$. Multiplication with $u=e^{2t^2}$ gives

$$(e^{2t^2}y)'=8te^{2t^2} \quad\Rightarrow\quad e^{2t^2}y=\int 8te^{2t^2}\,\mathrm{d}t$$

Using the substitution $v=2t^2$ and $\mathrm{d}v=4t\,\mathrm{d}t$, we compute the integral on the right-hand side:

$$\int 8te^{2t^2}\,\mathrm{d}t=\int 8te^v\frac{1}{4t}\,\mathrm{d}v=\int 2e^v\,\mathrm{d}v=2e^v+C=2e^{2t^2}+C$$

This gives the general solution

$$e^{2t^2}y=2e^{2t^2}+C \quad\Rightarrow\quad y=2+Ce^{-2t^2}$$

Alternatively, we could write the differential equation as $y'=8t-4ty=4t\cdot(2-y)$ and solve it as a separable differential equation.

(b) We write the difference equation in the form $(y^2-2t)+2yt\cdot y'=0$, and try to find a function $h=h(t,y)$ such that

$$h'_t=y^2-2t,\quad h'_y=2yt$$

We see that $h=y^2t-t^2+C(y)$ is the general solution of the first condition, and when we substitute this into the second condition, we find that $h'_y=2yt+C'(y)$, and therefore $h'_y=2yt$ when $C'(y)=0$. The simplest solution for h is therefore $h(t,y)=y^2t-t^2$ with $C(y)=0$. Since the differential equation is exact of the form $h'_t+h'_y\cdot y'=0$, the general solution is given by

$$h(t,y)=y^2t-t^2=C$$

The initial condition $y(1)=2$ gives $2^2\cdot 1-1^2=C$, or $C=3$. Hence the particular solution is

$$y^2t-t^2=3 \quad\Rightarrow\quad y^2=\frac{3+t^2}{t} \quad\Rightarrow\quad y=\sqrt{\frac{3+t^2}{t}}$$

We have chosen the positive square root in order for the particular solution to satisfy $y(1)=2$.

(c) The characteristic equation of A is $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} 2-\lambda & 0 \\ 1 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) = 0$$

and the two eigenvalues of A are therefore $\lambda_1 = 2$ and $\lambda_2 = -1$. Since each eigenvalue has multiplicity one, there is a base $\mathbf{v}_i$ for E_{λ_i} which we can find using Gaussian elimination:

$$E_2: \begin{pmatrix} 0 & 0 \\ 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \qquad E_{-1}: \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

We may choose the base vectors $\mathbf{v}_1 = (3, 1)$ and $\mathbf{v}_2 = (0, 1)$. We find the equilibrium state by solving $A\mathbf{y} + \mathbf{b} = \mathbf{0}$, which is a linear system $A\mathbf{y} = -\mathbf{b}$ that we can solve using Gaussian elimination:

$$\left(\begin{array}{cc|c} 2 & 0 & 2 \\ 1 & -1 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 2 & 0 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 2 & 4 \end{array}\right)$$

Back substitution gives $2y = 4$, or $y = 2$, and $x - 2 = -1$, or $x = 1$. The equilibrium state is therefore $(1, 2)$, and the general solution of the system of linear differential equations is

$$\mathbf{y} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot e^{2t} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot e^{-t}$$

(d) We can write the differential equation $y' + ty^2 = t$ as $y' = t - ty^2 = t \cdot (1 - y^2)$ and solve it as a separable differential equation. This gives

$$\frac{1}{1-y^2}\, y' = t \quad \Rightarrow \quad \int \frac{1}{1-y^2}\, \mathrm{d}y = \int t \, \mathrm{d}t \quad \Rightarrow \quad \int \frac{2}{1-y^2}\, \mathrm{d}y = \int 2t \, \mathrm{d}t$$

We have multiplied the equation with 2 for convenience. To compute the integral on the left-hand side, we use partial fractions and the factorization $1 - y^2 = (1 + y)(1 - y)$:

$$\frac{2}{1-y^2} = \frac{A}{1+y} + \frac{B}{1-y} \quad \Rightarrow \quad 2 = A(1-y) + B(1+y) = (A+B) + (-A+B)y$$

We have multiplied by the common denominator. We see that $A + B = 2$ and $B - A = 0$ by comparing coefficients, and this gives $A = B = 1$. This gives

$$\int \frac{1}{1+y}\, \mathrm{d}y + \int \frac{1}{1-y}\, \mathrm{d}y = \int 2t \, \mathrm{d}t \quad \Rightarrow \quad \ln|1+y| - \ln|1-y| = t^2 + C$$

and therefore we get

$$\ln \frac{|1+y|}{|1-y|} = t^2 + C \quad \Rightarrow \quad \left| \frac{1+y}{1-y} \right| = e^{t^2 + C} = e^{t^2} e^C \quad \Rightarrow \quad \frac{1+y}{1-y} = (\pm e^C) e^{t^2} = K e^{t^2}$$

We solve for y to get an explicit solution:

$$1 + y = (1-y)Ke^{t^2} \quad \Rightarrow \quad y(1 + Ke^{t^2}) = Ke^{t^2} - 1 \quad \Rightarrow \quad y = \frac{Ke^{t^2} - 1}{Ke^{t^2} + 1}$$