Solutions	Final exam in GRA 6035 Mathematics
Date	January 9th 2023 at 0900 - 1200

Question 1.

(a) We use cofactor expansion along the first row to find the determinant of A:

$$\begin{vmatrix} 12 & 6 & -3 \\ 6 & 3 & 6 \\ -3 & 6 & -4 \end{vmatrix} = 12(-12-36) - 6(-24+18) - 3(36+9) = -576 + 36 - 135 = -675$$

(b) Since A is a square matrix with determinant $|A| \neq 0$, it follows that A has maximal rank. Hence $\text{rk}(A) = 3$, and $\dim \text{Col}(A) = \text{rk}(A) = 3$.

(c) We determine when $\mathbf{v}$ is an eigenvector of A by computing $A\mathbf{v}$ and compare it with $\lambda\mathbf{v}$:

$$A\mathbf{v} = \begin{pmatrix} 12 & 6 & -3 \\ 6 & 3 & 6 \\ -3 & 6 & -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ a \end{pmatrix} = \begin{pmatrix} -3a \\ 6a \\ -15-4a \end{pmatrix}, \quad \lambda\mathbf{v} = \lambda \cdot \begin{pmatrix} 1 \\ -2 \\ a \end{pmatrix} = \begin{pmatrix} \lambda \\ -2\lambda \\ a\lambda \end{pmatrix}$$

We see that $A\mathbf{v} = \lambda\mathbf{v}$ if and only if $\lambda = -3a$, $-2\lambda = 6a$, and $a\lambda = -15-4a$. We substitute $\lambda = -3a$ in the last equation, and get $a(-3a) = -15-4a$, or $3a^2 - 4a - 15 = 0$. This gives

$$a = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(-15)}}{2(3)} = \frac{4 \pm \sqrt{196}}{6} = \frac{4 \pm 14}{6}$$

We get $a = 3$ and $a = -10/6 = -5/3$ as solutions to the quadratic equation. Hence $\mathbf{v}$ is in E_λ if and only if $a = 3,\ \lambda = -9$ or $a = -5/3,\ \lambda = 5$.

(d) Since A is a symmetric 3×3 matrix, it has three eigenvalue counted with multiplicities. We know from (c) that $\lambda_1 = -9$ and $\lambda_2 = 5$ are eigenvalues of A. Since $\text{tr}(A) = 12 + 3 - 4 = 11$, it follows that

$$\text{tr}(A) = 11 = \lambda_1 + \lambda_2 + \lambda_3 = -9 + 5 + \lambda_3 \quad \Rightarrow \quad \lambda_3 = 11 - (-9+5) = 15$$

It follows that the eigenvalues of A are $\lambda_1 = -9$, $\lambda_2 = 5$, and $\lambda_3 = 15$.

Question 2.

(a) The characteristic equation of the homogeneous difference equation $y_{t+2} - 2y_{t+1} - 8y_t = 0$ is $r^2 - 2r - 8 = 0$, which gives

$$r = \frac{2 \pm \sqrt{(-2)^2 - 4(-8)}}{2} = \frac{2 \pm \sqrt{36}}{2} = \frac{2 \pm 6}{2}$$

Hence there are two distinct characteristic roots $r_1 = 4$ and $r_2 = -2$, and the general homogeneous solution is

$$y_t^h = C_1 \cdot 4^t + C_2 \cdot (-2)^t$$

To find a particular solution of $y_{t+2} - 2y_{t+1} - 8y_t = -9t$, we use the method of undetermined coefficients with $y_t = At + B$, which gives $y_{t+1} = A(t+1) + B$ and $y_{t+2} = A(t+2) + B$. We get $(At + 2A + B) - 2(At + A + B) - 8(At + B) = -9t$ when we substitute this into the difference equation, and this implies that $(-9A)t + (-9B) = -9t$, or $A = 1$ and $B = 0$ by comparing coefficients. Hence $y_t^p = t$, and the general solution of the difference equation is

$$y_t = y_t^h + y_t^p = C_1 \cdot 4^t + C_2 \cdot (-2)^t + t$$

(b) The differential equation $y' = 2t - 4ty$ can be written as $y' = 2t(1-2y)$ and solved by separation of variables:

$$\frac{1}{1-2y} y' = 2t \quad \Rightarrow \quad \int \frac{1}{1-2y}\,\mathrm{d}y = \int 2t\,\mathrm{d}t \quad \Rightarrow \quad -\frac{1}{2}\ln|1-2y| = t^2 + C$$

This gives $\ln|1-2y| = -2t^2 - 2C$, or $|1-2y| = e^{-2t^2-2C}$, hence $1 - 2y = \pm e^{-2t^2-2C} = Ke^{-2t^2}$, or $2y = 1 - Ke^{-2t^2}$. This implies that the general solution is

$$y = \frac{1}{2} - \frac{K}{2} e^{-2t^2}$$

Alternatively, the differential equation can be solved using integrating factor by writing it in the standard form $y' + 4ty = 2t$ of a first order linear differential equation.

(c) We write $\mathbf{y} = (u, v)$ such that the system of differential equations can be written $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$ with

$$A = \begin{pmatrix} 0 & 1 \\ 8 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 24 \end{pmatrix}$$

The equilibrium state of the system is given by $A\mathbf{y} + \mathbf{b} = \mathbf{0}$, hence $v = 0$ and $8u + 2v + 24 = 0$. The solution is $v = 0$ and $8u = -24$, or $u = -3$, hence $\mathbf{y}_e = (-3, 0)$ is the equilibrium state. The eigenvalues of A are given by the characteristic equation $\lambda^2 - 2\lambda - 8 = 0$, and this gives $\lambda_1 = 4$ and $\lambda_2 = -2$. To find a base $\{\mathbf{v}_i\}$ for E_{λ_i} in each case, we use the Gaussian processes

$$E_4 : \begin{pmatrix} -4 & 1 \\ 8 & -2 \end{pmatrix} \to \begin{pmatrix} -4 & 1 \\ 0 & 0 \end{pmatrix} \qquad E_{-2} : \begin{pmatrix} 2 & 1 \\ 8 & 4 \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

and back substitution, and find base vectors $\mathbf{v}_1 = (1, 4)$ and $\mathbf{v}_2 = (1, -2)$ for the two eigenspaces. Hence the general solution of the system of differential equations is

$$\mathbf{y} = \begin{pmatrix} u \\ v \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} \cdot e^{4t} + C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdot e^{-2t} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

(d) The differential equation $ty' = e^{-y}$ can be written as $y' = e^{-y} \cdot 1/t$ and solved by separation of variables:

$$e^y y' = \frac{1}{t} \quad \Rightarrow \quad \int e^y \, \mathrm{d}y = \int \frac{1}{t} \, \mathrm{d}t \quad \Rightarrow \quad e^y = \ln(t) + C$$

We use $\ln|t| = \ln(t)$ since we assume that $t > 0$. The initial condition $y(1) = 0$ means that $t = 1, y = 0$ will be a solution, hence $e^0 = \ln(1) + C$, or $C = 1$. This gives $e^y = \ln(t) + 1$, and the particular solution with $y(1) = 0$ is given by $y = \ln(\ln(t) + 1)$.

Question 3.

(a) To determine the definiteness of the quadratic form f, we write down its symmetric matrix A:

$$A = \begin{pmatrix} 2 & -1 & 3 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 0 & 10 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Its leading principal minors are $D_1 = 2$, $D_2 = 2 - 1 = 1$, $D_3 = 3(-3) + 10D_2 = 10 - 9 = 1$, and $D_4 = |A|$ is given by

$$D_4 = -1 \begin{vmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix} + 1 \cdot D_3 = -1(1)(2-1) + 1 = 0$$

Since $D_1, D_2, D_3 > 0$ and $D_4 = 0$, we have that $\mathrm{rk}(A) = 3$ and A is positive semidefinite by the RRC. Hence f is a positive semidefinite quadratic form.

(b) The Lagrangian is $\mathcal{L} = \mathbf{x}^T A \mathbf{x} - \lambda(B\mathbf{x} - 33)$, where A is the symmetric matrix of f and $B = \begin{pmatrix} 3 & 1 & 8 & -4 \end{pmatrix}$. The Lagrange conditions are the first order conditions, which can be written $2A\mathbf{x} - \lambda B^T = \mathbf{0}$ in matrix form, and the constraint $B\mathbf{x} = 33$. We write the FOC's as $A\mathbf{x} = (\lambda/2)B^T$ and write $t = \lambda/2$ for simplicity. This gives a linear system with the following augmented matrix, and we solve it using Gaussian elimination:

$$\left(\begin{array}{cccc|c} 2 & -1 & 3 & 0 & 3t \\ -1 & 1 & 0 & 0 & t \\ 3 & 0 & 10 & 1 & 8t \\ 0 & 0 & 1 & 1 & -4t \end{array}\right) \to \left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & t \\ 0 & 1 & 3 & 0 & 5t \\ 0 & 3 & 10 & 1 & 11t \\ 0 & 0 & 1 & 1 & -4t \end{array}\right) \to \left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & t \\ 0 & 1 & 3 & 0 & 5t \\ 0 & 0 & 1 & 1 & -4t \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Hence w is a free variable, $z = -4t - w$, $y = -3z + 5t = 17t + 3w$, and $x = y - t = 16t + 3w$ by back substitution. With $w = -4$, we get the solution

$$(x, y, z, w) = (16t - 12, 17t - 12, -4t + 4, -4)$$

of the FOC's. When we substitute this into the constraint, we find that

$$3(16t - 12) + (17t - 12) + 8(-4t + 4) - 4(-4) = 33t = 33$$

which gives $t = 1$, and this implies that $\lambda/2 = 1$, or $\lambda = 2$. We conclude that there is a unique candidate point with $w = -4$ that satisfies the Lagrange conditions:

$$(x, y, z, w; \lambda) = (4, 5, 0, -4; 2)$$

(c) We apply the SOC to the candidate point $(x, y, z, w; \lambda) = (4, 5, 0, -4; 2)$ found in (b), and consider the function

$$h(x, y, z, w) = \mathcal{L}(x, y, z, w; 2) = \mathbf{x}^T A\mathbf{x} - 2(B\mathbf{x} - 33)$$

Since the Hessian $H(h) = 2A$ and A is positive semidefinite by (a), it follows that h is convex. Therefore, the candidate point is a minimum in the Lagrange problem by the SOC, and $f_{\min} = f(4, 5, 0, -4) = 2(4)^2 - 2(4)(5) + (5)^2 + (-4)^2 = 33$.

(d) We think of $p(x, y, z, w)$ as a composite function $p(u) = u^2 - 4u + 7$, with inner function or kernel $u = f(x, y, z, w)$. The inner function is convex with minimum value $u_{\min} = f(0, 0, 0, 0) = 0$ since f is a positive semidefinite quadratic form. The outer function $p(u) = u^2 - 4u + 7$ has derivative $p'(u) = 2u - 4 = 2(u - 2)$, hence it is decreasing for $0 \le u \le 2$ and increasing for $u \ge 2$, with $p(u) \to \infty$ when $u \to \infty$. Hence p has minimum value $p_{\min} = p(2) = 3$, and no maximum value. The range of p is $V_p = [3, \infty)$.