Solutions	Final exam in GRA 6035 Mathematics
Date	November 30th, 2022 at 1400 - 1700

Question 1.

(a) We use Gaussian elimination to find the rank of A:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are three pivot positions, we have that $\text{rk}(A) = 3$.

(b) Using the echelon form from (a), we see that $\dim \text{Null}(A) = 4 - 3 = 1$ and that w is a free variable. Back substitution gives that $-z + w = 0$, or $z = w$, that $-y + z + w = -y + 2w = 0$, or $y = 2w$, and that $x + y + z = x + 2w + w = 0$, or $x = -3w$. Hence $\text{Null}(A)$ consists of the vectors of the form $(x, y, z, w) = (-3w, 2w, w, w) = w \cdot \mathbf{w}$ with $\mathbf{w} = (-3, 2, 1, 1)$, and it follows that $\{\mathbf{w}\}$ is a base of $\text{Null}(A)$ with $\mathbf{w} = (-3, 2, 1, 1)$.

(c) We check if $\mathbf{v}$ is an eigenvector of A by computing $A\mathbf{v}$ and try to write the product as $\lambda\mathbf{v}$:

$$A\mathbf{v} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -2 \\ 2 \end{pmatrix}, \quad \lambda\mathbf{v} = \lambda \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda \\ -\lambda \\ \lambda \end{pmatrix}$$

Since $A\mathbf{v} = \lambda\mathbf{v}$ for $\lambda = 2$, it follows that $\mathbf{v}$ is an eigenvector of A with eigenvalue $\lambda = 2$.

(d) Let U be the set of all vectors that are orthogonal to the null space of A. Any vector $\mathbf{u}$ in U must be orthogonal to $\mathbf{w} = (-3, 2, 1, 1)$, since $\mathbf{w}$ is in $\text{Null}(A)$. When we put $\mathbf{u} = (x, y, z, w)$, this means that

$$\mathbf{u} \cdot \mathbf{w} = (x, y, z, w) \cdot (-3, 2, 1, 1) = -3x + 2y + z + w = 0$$

On the other hand, if $\mathbf{u}$ is orthogonal to $\mathbf{w}$, then it is also orthogonal to any vector $w \cdot \mathbf{w}$ in $\text{Null}(A)$, since $\mathbf{u} \cdot (w \cdot \mathbf{w}) = w(\mathbf{u} \cdot \mathbf{w}) = 0$. Hence U consists of all solutions of the homogeneous linear system

$$-3x + 2y + z + w = 0$$

It is clear that there are three free variables, and that we may take any three variables as free, for example x, y, z (to simplify computations). This gives $w = 3x - 2y - z$ with x, y, z free, and the vectors in U are given as

$$\mathbf{u} = (x, y, z, 3x - 2y - z) = x \cdot (1, 0, 0, 3) + y \cdot (0, 1, 0, -2) + z \cdot (0, 0, 1, -1)$$

Therefore $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a base of U, with $\mathbf{u}_1 = (1, 0, 0, 3),\ \mathbf{u}_2 = (0, 1, 0, -2),\ \mathbf{u}_3 = (0, 0, 1, -1)$. Alternatively, we could take y, z, w as free, and write $3x = 2y + z + w$, or $x = 2y/3 + z/3 + w/3$. Then the vectors in U can be written

$$\mathbf{u} = (2y/3 + z/3 + w/3, y, z, w) = y/3 \cdot (2, 3, 0, 0) + z/3 \cdot (1, 0, 3, 0) + w/3 \cdot (1, 0, 0, 3)$$

Therefore $\{\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3\}$ is also a base of U, with $\mathbf{u}'_1 = (2, 3, 0, 0),\ \mathbf{u}'_2 = (1, 0, 3, 0),\ \mathbf{u}'_3 = (1, 0, 0, 3)$.

Question 2.

(a) To determine the definiteness of the quadratic form q, we write down its symmetric matrix A:

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Its leading principal minors are $D_1 = 3$, $D_2 = 12 - 4 = 8$, and $D_3 = 1(2-4) - 1(3-2) + 1(8) = 5$. Since all leading principal minors of A are positive, q is a positive definite quadratic form.

(b) We can think of $p(x, y, z)$ as a composite function $p(u) = u \cdot e^u$, with inner function or kernel $u(\mathbf{x}) = 1 - q(\mathbf{x})$. We have that $H(u) = 0 - H(q) = -2A$ is negative definite, since A is positive definite from (a), and therefore u is concave with $u_{\max} = u(\mathbf{0}) = 1$ at $\mathbf{x} = \mathbf{0}$. In particular, the range of u is $V_u = (-\infty, 1]$. The outer function $p(u) = ue^u$ has derivative

$$p'(u) = 1 \cdot e^u + u \cdot e^u = (1 + u)e^u$$

This means that p in decreasing for $u \le -1$ and that p is increasing for $u \ge -1$. Since we consider this function for all $u \le 1$, we have that $p_{\min} = p(-1) = -1 \cdot e^{-1} = -1/e \cong -0.37$. Since $p(u) \to 0$ when $u \to -\infty$ and $p(1) = 1 \cdot e^1 = e > 0$, we see that $p_{\max} = p(1) = e \cong 2.71$.

(c) Since $(x^*, y^*, z^*; \lambda^*)$ satisfies the Lagrange conditions, we can apply the second order condition (SOC). We consider the function

$$h(x, y, z) = \mathcal{L}(x, y, z; \lambda^*) = x + y + z - \lambda^*(q(x, y, z) - 4)$$

with Hessian $H(h) = 0 - \lambda^* \cdot (2A - 0) = -2\lambda^* \cdot A$. Since A is positive definite, $H(h)$ is positive definite and h is convex if $\lambda^* < 0$, and $H(h)$ is negative definite and h is concave if $\lambda^* > 0$. By the SOC, we have that (x^*, y^*, z^*) is a minimum point when $\lambda^* < 0$ and a maximum point when $\lambda^* > 0$. Therefore statements (B) and (C) are true and statements (A) and (D) are false.

(d) Let B be the row vector $B = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ and let A be the symmetric matrix of the quadratic form q, given in (a). We can write $f(\mathbf{x}) = B\mathbf{x}$ and $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ in matrix form, and write the Lagrangian as $\mathcal{L} = B\mathbf{x} - \lambda(\mathbf{x}^T A\mathbf{x} - 4)$. Then the first order conditions (FOC) can be written $\mathcal{L}'(\mathbf{x}) = B^T - \lambda \cdot 2A\mathbf{x} = \mathbf{0}$, and the constraint can be written $\mathbf{x}^T A\mathbf{x} = 4$. From the FOC's, we get $2\lambda A\mathbf{x} = B^T$. Since $\lambda = 0$ does not give solutions, we can write this as

$$A\mathbf{x} = \frac{1}{2\lambda} \cdot B^T = s \cdot B^T = s \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} s \\ s \\ s \end{pmatrix}$$

with $s = 1/(2\lambda)$. **Alternative 1.** We use Gaussian elimination to solve the linear system:

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & s \\ 2 & 4 & 1 & s \\ 1 & 1 & 1 & s \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & s \\ 3 & 2 & 1 & s \\ 2 & 4 & 1 & s \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & s \\ 0 & -1 & -2 & -2s \\ 0 & 2 & -1 & -s \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & s \\ 0 & -1 & -2 & -2s \\ 0 & 0 & -5 & -5s \end{array}\right)$$

Back substitution gives $-5z = -5s$, or $z = s$, $-y - 2s = -2s$, or $y = 0$, and $x + 0 + s = s$, or $x = 0$. We find that $(x, y, z) = (0, 0, s)$, and the constraint gives $q(0, 0, s) = s^2 = 4$, or $s = \pm 2$, and $\lambda = 1/(2s) = \pm 1/4$. **Alternative 2.** We use that $|A| = D_3 = 5 \neq 0$ from (a), hence the matrix A is invertible, and we find that $\mathbf{x} = A^{-1}(sB^T) = sA^{-1}B^T$. We put this into the constraint $\mathbf{x}^T A\mathbf{x} = 4$, and find that

$$\mathbf{x}^T A\mathbf{x} = (sA^{-1}B^T)^T A(sA^{-1}B^T) = s^2(B(A^{-1})^T AA^{-1}B^T) = s^2(BA^{-1}B^T) = 4$$

We have used that A^{-1} is symmetric since A is symmetric, which means that $(A^{-1})^T = A^{-1}$. We compute A^{-1} using cofactors, and the fact that A is symmetric and therefore the cofactor matrix of A is symmetric. This gives the following expressions for A^{-1} and $BA^{-1}B^T$:

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 3 & -1 & -2 \\ -1 & 2 & -1 \\ -2 & -1 & 8 \end{pmatrix} \quad \Rightarrow \quad BA^{-1}B^T = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \cdot \frac{1}{5} \begin{pmatrix} 3 & -1 & -2 \\ -1 & 2 & -1 \\ -2 & -1 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} = \frac{5}{5} = 1$$

It follows that the constraint is $s^2 \cdot 1 = 4$, which gives $s = \pm 2$, and $\lambda = 1/(2s) = \pm 1/4$. Therefore

$$\mathbf{x} = sA^{-1}B^T = s \cdot \frac{1}{5} \begin{pmatrix} 3 & -1 & -2 \\ -1 & 2 & -1 \\ -2 & -1 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = s \cdot \frac{1}{5} \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix}$$

With either of the two methods, we find two candidate points $(0,0,2;1/4)$ with $f(0,0,2)=2$ and $(0,0,-2;-1/4)$ with $f(0,0,-2)=-2$. We use the results in (c), and find that $f_{\max}=2$ at $(x,y,z)=(0,0,2)$ with $\lambda=1/4$, and $f_{\min}=-2$ at $(x,y,z)=(0,0,-2)$ with $\lambda=-1/4$.

Question 3.

(a) The characteristic equation of the homogeneous differential equation $4y''+4y'-3y=0$ is $4r^2+4r-3=0$, which gives

$$r=\frac{-4\pm\sqrt{4^2-4\cdot 4(-3)}}{2\cdot 4}=\frac{-4\pm\sqrt{64}}{8}=\frac{-4\pm 8}{8}$$

Hence there are two distinct characteristic roots $r_1=4/8=1/2$ and $r_2=-12/8=-3/2$, and the general homogeneous solution is

$$y_h=C_1\,e^{t/2}+C_2\,e^{-3t/2}$$

To find a particular solution of $4y''+4y'-3y=8+8t-3t^2$, we use the method of undetermined coefficients with $y=At^2+Bt+C$, which gives $y'=2At+B$ and $y''=2A$. When we substitute this into the differential equation, we get $4(2A)+4(2At+B)-3(At^2+Bt+C)=8+8t-3t^2$, which gives $-3A=-3$, $8A-3B=8$, and $8A+4B-3C=8$ by comparing coefficients. Hence $A=1$, $B=0$, and $C=0$, which gives $y_p=t^2$. The general solution of the differential equation is

$$y=y_h+y_p=C_1\,e^{t/2}+C_2\,e^{-3t/2}+t^2$$

(b) We write $\mathbf{y}_t=(u_t,v_t)$ for $t=0,1,2,\dots$ such that the system of difference equations can be written $\mathbf{y}_{t+1}=A\mathbf{y}_t$ with

$$A=\begin{pmatrix}0.7 & 0.8\\ 0.4 & 0.3\end{pmatrix}$$

The eigenvalues of A are given by the characteristic equation $\lambda^2-\lambda-0.11=0$ since $\operatorname{tr}(A)=1$ and $\det(A)=0.21-0.32=-0.11$. This gives

$$\lambda=\frac{1\pm\sqrt{(-1)^2-4(-0.11)}}{2}=\frac{1\pm\sqrt{1.44}}{2}=\frac{1\pm 1.2}{2}$$

and the two eigenvalues are $\lambda_1=1.1$ and $\lambda_2=-0.1$. To find a base $\{\mathbf{v}_i\}$ for E_{λ_i} in each case, we use the Gaussian processes

$$E_{1.1}:\begin{pmatrix}-0.4 & 0.8\\ 0.4 & -0.8\end{pmatrix}\to\begin{pmatrix}-1 & 2\\ 0 & 0\end{pmatrix}\qquad\qquad E_{-0.1}:\begin{pmatrix}0.8 & 0.8\\ 0.4 & 0.4\end{pmatrix}\to\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$$

and back substitution, and find base vectors $\mathbf{v}_1=(2,1)$ and $\mathbf{v}_2=(-1,1)$ for the two eigenspaces. Hence the general solution of the system of difference equations is

$$\mathbf{y}_t=\begin{pmatrix}u_t\\ v_t\end{pmatrix}=C_1\begin{pmatrix}2\\ 1\end{pmatrix}\cdot 1.1^t+C_2\begin{pmatrix}-1\\ 1\end{pmatrix}\cdot(-0.1)^t=\begin{pmatrix}2C_1\cdot 1.1^t-C_2(-0.1)^t\\ C_1\cdot 1.1^t+C_2(-0.1)^t\end{pmatrix}$$

(c) We write $2ty^2-4y+(2t^2y-4t)y'=0$ as $p+q\cdot y'=0$, and see that this differential equation is exact if there is a function $h=h(t,y)$ such that $h_t'=p=2ty^2-4y$ and $h_y'=q=2t^2y-4t$. From the first condition, we obtain $h=t^2y^2-4ty+C(y)$, and substituting this into the second condition, we get $h_y'=2t^2y-4t+C'(y)=2t^2y-4t$. We see that $C(y)=0$ gives the solution $h=t^2y^2-4ty$. This means that the differential equation is exact, and that its general solution is given by $t^2y^2-4ty=C$. The initial condition $y(1)=5$ gives $1^2\cdot 5^2-4\cdot 1\cdot 5=C$, or $C=5$. This gives

$$t^2y^2-4ty=5\quad\Rightarrow\quad t^2y^2-4ty-5=(ty+1)(ty-5)=0$$

The explicit solutions are therefore $y=-1/t$ or $y=5/t$, and since we have $y(1)=5$, the particular solution is $y=5/t$.

(d) We use the hint to find homogeneous solutions: If $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. We substitute these expressions into the homogeneous equation $t^2y'' + 4ty' + 2y = 0$:

$$t^2(r(r-1)t^{r-2}) + 4t(rt^{r-1}) + 2(t^r) = (r^2 - r + 4r + 2)t^r = (r^2 + 3r + 2)t^r = 0$$

This means that $y = t^r$ is a homogeneous solution if and only if $r^2 + 3r + 2 = 0$, which gives $r = -1$ and $r = -2$. The general homogeneous solution is therefore

$$y_h = C_1 \cdot t^{-1} + C_2 \cdot t^{-2} = \frac{C_1 \cdot t + C_2}{t^2}$$

We use the method of undetermined coefficients to find a particular solution, and use $y = A$ since the right-hand side of $t^2y'' + 4ty' + 2y = 6$ is a constant. If $y = A$, then $y' = y'' = 0$, and when we substitute this into the differential equation, we get $t^2(0) + 4t(0) + 2(A) = 6$, or $A = 3$. Hence $y_p = 3$ and the general solution of the differential equation is

$$y = y_h + y_p = \frac{C_1 \cdot t + C_2}{t^2} + 3 = \frac{C_1 \cdot t + C_2 + 3t^2}{t^2}$$