Exam	Final exam in GRA 6035 Mathematics
Date	April 29th, 2022 at 1300 - 1600

This exam consists of 12+1 problems (one additional problem is for extra credits, and can be skipped). Each problem has a maximal score of 6p, and 72p (12 solved problems) is marked as 100% score.

You must give reasons for your answers. Precision and clarity will be emphasized when evaluating your answers.

Question 1.

We consider the matrix A given by

$$A = \begin{pmatrix} 2 & -4 & -11 \\ -2 & 3 & 10 \\ 1 & -4 & -10 \end{pmatrix}$$

(a) **(6p)** Compute the determinant and rank of A.
(b) **(6p)** Find a base of $\text{Null}(A - I)$.
(c) **(6p)** Show that $\lambda = 1, \lambda = -5$ and $\lambda = -1$ are eigenvalues of A.
(d) **(6p)** Find a matrix P such that $P^{-1}AP$ is diagonal, if it is possible.

Question 2.

(a) **(6p)** Solve the differential equation $y'' - 3y' + 2y = 6e^{-t}$.
(b) **(6p)** Solve the differential equation $ty' + y = 1$.
(c) **(6p)** Solve the differential equation $2ty' + y^2 = 1$.
(d) **(6p)** Find the equilibrium state of the system of differential equations. Is it stable?

$$\mathbf{y}' = \begin{pmatrix} 2 & -4 & -11 \\ -2 & 3 & 10 \\ 1 & -4 & -10 \end{pmatrix} \cdot \mathbf{y} + \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$$

Question 3.

We consider the function f given by $f(x, y, z) = 4y - 2x^2 - 3z^2 - 2xy - 8xz$ and the Lagrange problem given by

$$\max\ f(x, y, z) \text{ subject to } g(x, y, z) = x^2 + y^2 + 4z^2 + 4yz = 2$$

(a) **(6p)** Find all stationary points of f and classify them.
(b) **(6p)** Write down the Lagrange conditions of the Lagrange problem.
(c) **(6p)** Find all points $(x, y, z; \lambda)$ with $\lambda = 1$ that satisfy the Lagrange conditions.
(d) **(6p)** Solve the Lagrange problem.
(e) **Extra credit (6p)** Find a linear change of variables $\mathbf{x} = P\mathbf{u}$ such that the constraint $g(\mathbf{x}) = 2$ takes the form $\lambda_1 u_1^2 + \lambda_2 u_2^2 + \lambda_3 u_3^2 = 2$, and use this to describe the set of admissible points.