Solutions	Final exam in GRA 6035 Mathematics
Date	January 28th, 2022 at 1300 - 1600

Question 1.

(a) We compute the determinant of A using cofactor expansion along the second column:

$$|A| = \begin{vmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 4 & 4 \\ -1 & 0 & -2 & -5 \\ 1 & 0 & 0 & 3 \end{vmatrix} = 2 \cdot \begin{vmatrix} 3 & 0 & 1 \\ -1 & -2 & -5 \\ 1 & 0 & 3 \end{vmatrix}$$

We compute the 3-minor by cofactor expansion along the middle column, and this gives

$$|A| = 2 \cdot (-2) \cdot (9-1) = -4 \cdot 8 = -32$$

(b) Since A is a 4×4 matrix with $|A| \neq 0$, we have that $\text{rk}(A) = 4$. This means that

$$\dim \text{Col}(A) = \text{rk}(A) = 4, \quad \dim \text{Null}(A) = 4 - \text{rk}(A) = 0$$

(c) We solve the linear system $(A - 2I)\mathbf{x} = \mathbf{0}$, to simultaneously check that $\lambda = 2$ is an eigenvalue and to find a base of E_2:

$$A - 2I = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ -1 & 0 & -4 & -5 \\ 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that there are two free variables x_2 and x_4, hence $\lambda = 2$ is an eigenvalue and $\dim E_2 = 2$. We solve the linear system using back substitution, and find $x_3 = -x_4$ and $x_1 = -x_4$. The solutions are therefore given by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

It follows that the vectors $\mathbf{v}_1 = (0, 1, 0, 0), \ \mathbf{v}_2 = (-1, 0, -1, 1)$ form a base of E_2.

(d) The eigenvalues of A are the solutions of the characteristic equation $|A - \lambda I| = 0$, and we compute the determinant on the left-hand side by cofactor expansion along the second column:

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 & 1 \\ 0 & 2-\lambda & 4 & 4 \\ -1 & 0 & -2-\lambda & -5 \\ 1 & 0 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda) \cdot \begin{vmatrix} 3-\lambda & 0 & 1 \\ -1 & -2-\lambda & -5 \\ 1 & 0 & 3-\lambda \end{vmatrix}$$

We compute the resulting 3-minor using cofactor expansion along the second column, and write the characteristic equation in the form

$$(2-\lambda) \cdot \begin{vmatrix} 3-\lambda & 0 & 1 \\ -1 & -2-\lambda & -5 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) \cdot \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

This gives $\lambda = 2$, $\lambda = -2$, or $\lambda^2 - 6\lambda + 8 = 0$, which gives $\lambda = 2$ or $\lambda = 4$. We conclude that the eigenvalues of A are $\lambda_1 = \lambda_2 = 2, \ \lambda_3 = 4, \ \lambda_4 = -2$.

Question 2.

(a) We use superposition to solve the linear difference equation $6y_{t+2} + y_{t+1} - y_t = 6t + 1$: To find the homogeneous solution y_t^h, we consider the characteristic equation $6r^2 + r - 1 = 0$, with two distinct roots $r = 1/3, \ r = -1/2$, and therefore

$$y_t^h = C_1 \left(\frac{1}{3}\right)^t + C_2 \left(-\frac{1}{2}\right)^t$$

To find a particular solution y_t^p, we consider the difference equation $6y_{t+2} + y_{t+1} - y_t = 6t + 1$. We try to find a constant solution $y_t = At + B$, which gives $y_{t+1} = A(t+1) + B = At + A + B$, and $y_{t+2} = A(t+2)+B = At+2A+B$. When we substitute this into the difference equation, we get

$$\begin{aligned}6(At+2A+B)+(At+A+B)-(At+B) &= 6t+1\\ (6A)t+(13A+6B) &= 6t+1\end{aligned}$$

Comparing coefficients, we find $6A = 6$, or $A = 1$, and $13A+6B = 1$, or $6B = 1-13 = -12$, which gives $B = -2$. The general solution is therefore given by

$$y_t = y_t^h + y_t^p = C_1\left(\frac{1}{3}\right)^t + C_2\left(-\frac{1}{2}\right)^t + t - 2$$

(b) The differential equation $ty' - 2y = t^2$ can be written $y' - (2/t)y = t$, and is linear. Since $\int -(2/t)\mathrm{d}t = -2\ln|t| + C$, the integrating factor is $u = e^{-2\ln|t|} = |t|^{-2} = 1/t^2$. Multiplication with the integrating factor gives the differential equation

$$\left(\frac{1}{t^2}\cdot y\right)' = \frac{1}{t} \quad\Rightarrow\quad \frac{1}{t^2}\cdot y = \int 1/t\,\mathrm{d}t = \ln|t| + C$$

Therefore, the differential equation has general solution $y = t^2\cdot\ln|t| + Ct^2$.

(c) We write $y^2 - 3t^2y + (2ty - t^3)y' = 0$ in the form $p(t,y) + q(t,y)y' = 0$ to check if it is exact: We look for a function $h(t,y)$ in two variables such that

$$\begin{aligned}h'_t &= p(t,y) = y^2 - 3t^2y\\ h'_y &= q(t,y) = 2ty - t^3\end{aligned}$$

We see that $h(t,y) = y^2t - t^3y$ satisfies both conditions, therefore the differential equation is exact, and its general solution can be written in the form $h(t,y) = y^2t - t^3y = C$. The implicit form of the solution can be written $ty^2 - t^3y - C = 0$, and we use the quadratic formula to solve it for y:

$$y = \frac{t^3 \pm \sqrt{t^6 - 4t(-C)}}{2t} = \frac{t^2}{2} \pm \frac{\sqrt{t^6 + 4Ct}}{2t} = \frac{1}{2}\left(t^2 \pm \sqrt{t^4 + 4C/t}\right)$$

(d) We let A be the 3×3 matrix such that the system of difference equations can be written in the form $\mathbf{y}_{t+1} = A\mathbf{y}_t$. The eigenvalues of A is given the characteristic equation

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1\\ -1 & 2-\lambda & 0\\ 3 & -1 & 1-\lambda\end{vmatrix} = 0$$

We use cofactor expansion along the second row to compute the determinant, and get

$$+1\cdot(1-\lambda+1) + (2-\lambda)((2-\lambda)(1-\lambda)-3) = (2-\lambda)(1+\lambda^2-3\lambda+2-3)$$

This gives the characteristic equation $(2-\lambda)(\lambda^2-3\lambda) = -\lambda(\lambda-2)(\lambda-3) = 0$, and there are three distinct eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2$, and $\lambda_3 = 3$. This means that A is diagonalizable. We find a base $\{\mathbf{v}_i\}$ for E_{λ_i} in each case: We use the Gaussian processes

$$E_0 : \begin{pmatrix} 2 & 1 & 1\\ -1 & 2 & 0\\ 3 & -1 & 1\end{pmatrix} \to \begin{pmatrix} -1 & 2 & 0\\ 0 & 5 & 1\\ 0 & 0 & 0\end{pmatrix} \qquad E_2 : \begin{pmatrix} 0 & 1 & 1\\ -1 & 0 & 0\\ 3 & -1 & -1\end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}$$

$$E_3 : \begin{pmatrix} -1 & 1 & 1\\ -1 & -1 & 0\\ 3 & -1 & -2\end{pmatrix} \to \begin{pmatrix} -1 & 1 & 1\\ 0 & 2 & 1\\ 0 & 0 & 0\end{pmatrix}$$

and back substitution, and find base vectors $\mathbf{v}_1 = (-2,-1,5)$, $\mathbf{v}_2 = (0,-1,1)$, $\mathbf{v}_3 = (1,-1,2)$ for the three eigenspaces. The general solution is therefore given by

$$\mathbf{y}_t = C_1\mathbf{v}_1\lambda_1^t + C_2\mathbf{v}_2\lambda_2^t + C_3\mathbf{v}_3\lambda_3^t = C_1\begin{pmatrix}-2\\-1\\5\end{pmatrix}\cdot 0^t + C_2\begin{pmatrix}0\\-1\\1\end{pmatrix}\cdot 2^t + C_3\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot 3^t$$

where $0^t = 1$ if $t = 0$ and $0^t = 0$ if $t > 0$ is a positive integer.

Question 3.

(a) We write f on matrix form $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + B\mathbf{x}$, where

$$A = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & -2 \\ 0 & -2 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 6 & 6 \end{pmatrix}$$

To find the stationary points, we solve the first order conditions $f'(\mathbf{x}) = 2A\mathbf{x} + B^T = \mathbf{0}$, or $A\mathbf{x} = -1/2 \cdot B^T$. This gives a linear system, and we solve it using Gaussian elimination:

$$\left(\begin{array}{ccc|c} -1 & -1 & 0 & -3 \\ -1 & 0 & -2 & -3 \\ 0 & -2 & -4 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & -1 & 0 & -3 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & -4 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & -1 & 0 & -3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -8 & -3 \end{array}\right)$$

Back substitution gives $-9z = -3$ or $z = 3/8$, $y = 2(3/8) = 3/4$, and that $-x = 3/4 - 3 = -9/4$ or $x = 9/4$. Hence $\mathbf{x}^* = (9/4, 3/4, 3/8)$ is the unique stationary point f. To classify it, notice that A is indefinite since $D_2 = -1$. This means that $H(f)(\mathbf{x}^*) = 2A$ is also indefinite, and $\mathbf{x}^* = (9/4, 3/4, 3/8)$ is a saddle point for f by the second derivative test.

(b) The Kuhn-Tucker problem is in standard form with Lagrangian $\mathcal{L} = \mathbf{x}^T A\mathbf{x} + B\mathbf{x} - \lambda(\mathbf{x}^T D\mathbf{x} - 9)$, where D is the symmetric matrix of the quadratic form g, given by

$$D = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & 3 \end{pmatrix}$$

The first order conditions (FOC) can therefore be written $\mathcal{L}'(\mathbf{x}) = 2A\mathbf{x} + B^T - \lambda(2D\mathbf{x}) = \mathbf{0}$, the constraint (C) can be written $\mathbf{x}^T D\mathbf{x} \le 9$, and the complementary slackness conditions can be written $\lambda \ge 0$ and $\lambda(\mathbf{x}^T D\mathbf{x} - 9) = 0$. Together, the conditions FOC + C + CSC are the Kuhn-Tucker conditions of the problem:

$$\text{FOC+C+CSC:} \quad 2A\mathbf{x} + B^T - \lambda(2D\mathbf{x}) = \mathbf{0},\ \mathbf{x}^T D\mathbf{x} \le 9,\ \lambda \ge 0,\ \lambda(\mathbf{x}^T D\mathbf{x} - 9) = 0$$

(c) When $\lambda = 1$, the first order conditions are $2A\mathbf{x} + B^T - 2D\mathbf{x} = \mathbf{0}$, or $(A - D)\mathbf{x} = -1/2 \cdot B^T$. This is a linear system, and we solve it using Gaussian elimination (where the first step is to subtract the last row from the first to simplify computations):

$$\left(\begin{array}{ccc|c} -3 & -1 & -4 & -3 \\ -1 & -1 & -2 & -3 \\ -4 & -2 & -7 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ -1 & -1 & -2 & -3 \\ -4 & -2 & -7 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 2 & 5 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 5 & -3 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Back substitution gives $z = -3$, $2y = -5(-3) - 3 = 12$ or $y = 6$, and that $x = -(6) - 3(-3) = 3$. We get the solution $(x, y, z; \lambda) = (3, 6, -3; 1)$ of the FOC. We see that the constraint is binding since

$$g(3, 6, -3) = 2(3)^2 + 6^2 + 3(-3)^2 + 8(3)(-3) = 9$$

at this point, and that the CSC is satisfied since $\lambda > 0$. We conclude that there is one candidate point $(x, y, z; \lambda) = (3, 6, -3; 1)$ with $\lambda = 1$ that satisfies the Kuhn-Tucker conditions.

(d) We use the second order condition (SOC) to test the candidate point $(x, y, z; \lambda) = (3, 6, -3; 1)$, and therefore consider the function

$$h(\mathbf{x}) = \mathcal{L}(\mathbf{x}; 1) = 2A\mathbf{x} + B^T - 2D\mathbf{x} = 2(A - D)\mathbf{x} + B^T$$

We notice that h is a quadratic function with Hessian $H(h) = 2(A - D)$, and that $H(h)$ has the same definiteness as

$$A - D = \begin{pmatrix} -3 & -1 & -4 \\ -1 & -1 & -2 \\ -4 & -2 & -7 \end{pmatrix}$$

We compute the principal minors of $A - D$: We have $D_1 = -3$, $D_2 = 3 - 1 = 2$, and that $D_3 = -3(7 - 4) + 1(7 - 8) - 4(2 - 4) = -2$. We conclude that $A - D$, and therefore $H(h) = 2(A - D)$, is negative definite, and it follows that h is a concave function. By the SOC, it follows that $(x, y, z) = (3, 6, -3)$ is a maximizer in the Kuhn-Tucker problem, and that $f_{\max} = f(3, 6, -3) = 27$ is the maximum value.

(e) We have that $g(x, y, z)$ can be written as a sum of the quadratic forms $2x^2 + 3z^2 + 8xz$ and y^2. We see that the second one is positive definite, while the first is indefinite. This means that D is not bounded, and therefore not compact. For example, we can let $y = 0$ and $z = -x$. Then the constraint

$$g(x, y, z) = 2x^2 + 3(-x)^2 + 8x(-x) = 5x^2 - 8x^2 = -3x^2 \leq 9$$

is satisfied for all values of x, and this means that there in no upper or lower bound on x for admissible points (x, y, z) in D.