Solutions	Final exam in GRA 6035 Mathematics
Date	March 19th, 2021 at 0900 - 1215

Question 1.

(a) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be the column vectors of A. We use elementary row operations to find an echelon form of A:

$$\begin{pmatrix} 1 & -1 & 0 & 4 \\ 3 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & -2 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 4 \\ 0 & 5 & 1 & -12 \\ 0 & 3 & 1 & -8 \\ 0 & -2 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 3 & 1 & -8 \\ 0 & -2 & 0 & 4 \end{pmatrix}$$

In the last step, we added 2 times the last row to the second row to simplify the computation. Then we continue the Gaussian process:

$$\begin{pmatrix} 1 & -1 & 0 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 2 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see from the pivot positions in the echelon form that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are not linearly independent vectors. We solve the linear system $A\mathbf{x} = \mathbf{0}$ to find a linear dependency relation: We see from the echelon form that w is free, and back substitution gives that $-2z + 4w = 0$, or $z = 2w$, that $y + z - 4w = 0$, or $y = -2w + 4w = 2w$, and that $x - y + 4w = 0$, or $x = 2w - 4w = -2w$. Hence the solutions are $\mathbf{x} = w(-2, 2, 2, 1)$ and $w = 1$ gives

$$-2\mathbf{v}_1 + 2\mathbf{v}_2 + 2\mathbf{v}_3 + \mathbf{v}_4 = 0 \quad \Rightarrow \quad \mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2 - \frac{1}{2}\mathbf{v}_4$$

(b) From (a), we see that $\operatorname{rk} A = 3$, hence $\dim \operatorname{Null}(A) = 4 - 3 = 1$. Since we have that

$$A \cdot \mathbf{w} = \begin{pmatrix} 1 & -1 & 0 & 4 \\ 3 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & -2 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \\ -2 \end{pmatrix} \neq \mathbf{0}$$

it follows that the vector $\mathbf{w}$ is not in $\operatorname{Null}(A)$.

(c) We get $f(\mathbf{x}) = x^2 + 2xy + 2xz + 4xw + 2y^2 + 2yz - 2yw + z^2 + 4w^2$ by multiplying the matrices when we write $\mathbf{x} = (x, y, z, w)$. We see that this is a quadratic form with symmetric matrix

$$B = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 4 \end{pmatrix}$$

To determine the definiteness of B, we compute its leading principal minors: We have $D_1 = 1$, $D_2 = 1$, $D_3 = 0$ (since the submatrix has two equal columns), and by cofactor expansion along the last row, we get

$$D_4 = |B| = -2 \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} + (-1) \cdot 0 + 4 \cdot 0 = -2(1(-1-2) - 1(-1-4)) = -4$$

Since $D_4 < 0$, B is indefinite. In particular, $|B| \neq 0$, and the stationary points are given by $2B\mathbf{x} = \mathbf{0}$, or $B\mathbf{x} = \mathbf{0}$. Therefore, the trivial solution $\mathbf{x} = \mathbf{0}$ is the unique stationary point, and it is a saddle point since B is indefinite.

(d) This is not true. When $\mathbf{x}$ is a eigenvector of M with eigenvalue λ, we have

$$f(\mathbf{x}) = \mathbf{x}^T M \mathbf{x} = \mathbf{x}^T \lambda \mathbf{x} = \lambda \mathbf{x}^T \mathbf{x} = \lambda \|\mathbf{x}\|^2 > 0$$

but in general, this will not work when we have a linear combination of several eigevectors. A counterexample is

$$M = \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$$

which has two positive eigenvalues $\lambda = 1, 2 > 0$ but the function $f(\mathbf{x}) = \mathbf{x}^T M \mathbf{x} = x^2 + 6xy + 2y^2$ is not positive definite since $f(-1, 1) = 1 - 6 + 2 = -3 < 0$.

Question 2.

(a) We have that $u = u(x, y, z) = 2 + Q(x, y, z)$ for the quadratic form Q with symmetric matrix

$$A = \begin{pmatrix} 7 & 4 & 2 \\ 4 & 13 & -1 \\ 2 & -1 & 1 \end{pmatrix}$$

We compute the leading principal minors $D_1 = 7$, $D_2 = 7 \cdot 13 - 4^2 = 91 - 16 = 75$, and $D_3 = |A| = 2(-4 - 26) - (-1)(-7 - 8) + 1(75) = -60 - 15 + 75 = 0$. Hence A is positive semi-definite by the RRC. Since $Q(\mathbf{x}) \geq 0$ is positive semidefinite, u is a convex function with minimum value $u_{\text{min}} = 2$. Since the outer function $f(u) = \ln(u)/u^3$ has derivative

$$f'(u) = \frac{1/u \cdot u^3 - \ln(u) \cdot 3u^2}{u^6} = \frac{1 - 3\ln(u)}{u^4}$$

it has a stationary point at $u = e^{1/3} = \sqrt[3]{e} \approx 1.40$, and $f' < 0$ for $u \geq \sqrt[3]{e}$. For $u \geq 2$, the outer function is decreasing, and this means that $f_{\text{max}} = f(2) = \ln(2)/8$ at $u = 2$. The maximum is attained at all points in $\text{Null}(A)$. For instance, $f(0, 0, 0) = \ln(2)/8$ since $u(0, 0, 0) = 2$.

(b) From the constraint $x^2 + y^2 + z^2 \leq 5$, it follows that $-\sqrt{5} \leq x, y, z \leq \sqrt{5}$, hence the set D of admissible points is closed and bounded, and therefore D is compact. If the constraint $x^2 + y^2 + z^2 = 5$ is binding, then the Jacobian matrix

$$J = \begin{pmatrix} 2x & 2y & 2z \end{pmatrix}$$

has maximal rank $\text{rk}\, J = 1$ since at least one of the variables must be non-zero, and in the non-binding case there is no NDCQ condition. Hence the NDCQ is satisfied for all admissible points.

(c) The Kuhn-Tucker problem is in standard form. Since $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$, where A is matrix in (a), and the constraint can be written $\mathbf{x}^T I \mathbf{x} \leq 5$, we have the Lagrangian

$$\mathcal{L} = \mathbf{x}^T A \mathbf{x} - \lambda \mathbf{x}^T I \mathbf{x} = \mathbf{x}^T (A - \lambda I) \mathbf{x}$$

This implies that the first order conditions can be written $2(A - \lambda I)\mathbf{x} = 0$. We can also see this by computing the first order conditions without using matrices. The solutions $(\mathbf{x}; \lambda)$ of the first order conditions are either points where $\mathbf{x} = \mathbf{0}$, or points $(\mathbf{x}; \lambda)$ where $\mathbf{x}$ is a non-zero eigenvector of A with eigenvalue λ. If $\mathbf{x} = \mathbf{0}$, then the constraint is non-binding by the CSC, and $\lambda = 0$, and $(0, 0, 0, 0; 0)$ is one candidate points with $Q = 0$. When $\mathbf{x}$ is a non-zero eigenvector with eigenvalue λ, then

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = \mathbf{x}^T (\lambda \mathbf{x}) = \lambda \mathbf{x}^T \mathbf{x} \leq 5\lambda$$

since $\mathbf{x}^T \mathbf{x} \leq 5$ by the constraint. We compute the eigenvalues of A:

$$|A - \lambda I| = \begin{vmatrix} 7 - \lambda & 4 & 2 \\ 4 & 13 - \lambda & -1 \\ 2 & -1 & 1 - \lambda \end{vmatrix} = -\lambda^3 + 21\lambda^2 - 90\lambda = 0$$

This gives eigenvalues $\lambda = 0, 6, 15$, and $Q(\mathbf{x}) \leq 5 \cdot 15 = 75$ since $\lambda = 15$ is the maximal eigenvalue. We find candidate points with $\lambda = 15$:

$$A - 15I = \begin{pmatrix} -8 & 4 & 2 \\ 4 & -2 & -1 \\ 2 & -1 & -14 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 2 & -1 & -14 \\ 0 & 0 & 27 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the eigenvectors in E_{15} are $\mathbf{x} = x(1, 2, 0) = (x, 2x, 0)$ with x free. Since $\lambda > 0$, the constraint is binding, and this gives $x^2 + (2x)^2 + 0^2 = 5x^2 = 5$, or $x = \pm 1$. Hence there are two candidate points $(1, 2, 0; 15)$, $(-1, -2, 0; 15)$ with $\lambda = 15$ and $Q = 75$. We use the SOC to check that these are maximum points:

$$h(\mathbf{x}) = \mathcal{L}(\mathbf{x}; 15) = \mathbf{x}^T (A - 15I) \mathbf{x}$$

Since A has eigenvalues $\lambda = 0, 6, 15$, $A - 15I$ has eigenvalues $\lambda = -15, -9, 0$ and is negative semidefinite. It follows that h is concave, and $Q_{\max} = 75$ at $(1, 2, 0)$ and $(-1, -2, 0)$ with $\lambda = 15$.

(d) We consider the Kuhn-Tucker problem with parameter a given by

$$\max ax^2 + 8xy + 4xz + 13y^2 - 2yz + z^2 \text{ when } x^2 + y^2 + z^2 \leq 5$$

From (c) we know that $Q^*(7) = 75$ when $a = 7$, and $\mathbf{x}^*(7) = (\pm 1, \pm 2, 0)$ with $\lambda^*(7) = 15$, and $\mathcal{L}'_a = x^2$. By the Envelope Theorem, it follows that

$$\frac{dQ^*(a)}{da} = \mathcal{L}'_a(\mathbf{x}^*(a); \lambda^*(a)) = x^*(a)^2 = (\pm 1)^2 = 1$$

at $a = 7$. This means that the maximum value for $a = 8$ can be estimated as

$$Q^*(8) \approx Q^*(7) + (8 - 7) \cdot 1 = 75 + 1 = 76$$

Question 3.

(a) The second order difference equation $y_{t+2} - 7y_{t+1} + 6y_t = -4 \cdot 2^t$ has characteristic equation $r^2 - 7r + 6 = 0$, with characteristic roots $r = 1$ and $r = 6$. The homogeneous solution is therefore $y_t^h = C_1 \cdot 1^t + C_2 \cdot 6^t = C_1 + C_2 \cdot 6^t$. To find a particular solution, we use $y_t = A \cdot 2^t$ since $f_t = -4 \cdot 2^t$. This gives

$$y_{t+2} - 7y_{t+1} + 6y_t = 4A \cdot 2^t - 14A \cdot 2^t + 6A \cdot 2^t = -4A \cdot 2^t = -4 \cdot 2^t$$

Hence $-4A = -4$, or $A = 1$. The general solution is therefore $y_t = C_1 + C_2 \cdot 6^t + 2^t$. We have $y_1 = C_1 + 6C_2 + 2 = 9$ and $y_3 = C_1 + 216C_2 + 8 = 225$. This gives $C_1 + 6C_2 = 7$ and $C_1 + 216C_2 = 217$. When we subtract the equations, we get $210C_2 = 210$, or $C_2 = 1$, and it follows that $C_1 = 1$. The solution is $y_t = 1 + 6^t + 2^t$.

(b) To solve $y' + y - 1 = t(y - 1)$ as a linear differential equation, we write it as $y' + (1 - t)y = 1 - t$. Since $\int 1 - t\,dt = t - t^2/2 + C$, we can use the integrating factor $u = e^{t - t^2/2}$, and this gives

$$(yu)' = (1 - t)e^{t - t^2/2} \quad \Rightarrow \quad yu = \int (1 - t)e^{t - t^2/2}\,dt = e^{t - t^2/2} + C$$

This gives the general solution $y = 1 + Ce^{t^2/2 - t}$. To solve $y' + y - 1 = t(y - 1)$ as a separable differential equation, we write it as $y' = t(y - 1) - (y - 1) = (t - 1)(y - 1)$. This gives

$$\frac{1}{y - 1} y' = t - 1 \quad \Rightarrow \quad \ln|y - 1| = \int t - 1\,dt = t^2/2 - t + C$$

This gives $|y - 1| = e^{t^2/2 - t + C}$, or $y - 1 = Ke^{-t + t^2/2}$ with $K = \pm e^C$. We find the general solution $y = 1 + Ke^{-t + t^2/2}$. If $y(0) = 4$, then $1 + K = 4$, or $K = 3$, and $y(2) = 1 + 3e^{2^2/2 - 2} = 1 + 3e^0 = 4$.

(c) The eigenvalues of A are given by the characteristic equation

$$|A - \lambda I| = \begin{vmatrix} 4 - \lambda & -1 & 2 \\ 1 & 1 - \lambda & -1 \\ 2 & -1 & 4 - \lambda \end{vmatrix} = 0$$

Cofactor expansion along the first row gives

$$\begin{aligned} |A - \lambda I| &= (4 - \lambda)((1 - \lambda)(4 - \lambda) - 1) - (-1)(4 - \lambda + 2) + 2(-1 - 2(1 - \lambda)) = 0 \\ &= (1 - \lambda)(4 - \lambda)^2 - (4 - \lambda) + 6 - \lambda + 4\lambda - 6 \\ &= (1 - \lambda)(4 - \lambda)^2 + 4\lambda - 4 = (1 - \lambda)(\lambda^2 - 8\lambda + 12) = -(\lambda - 1)(\lambda - 2)(\lambda - 6) \end{aligned}$$

Hence A has three distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 6$, and we find an eigenvector $\mathbf{v}_i$ for λ_i for $1 \leq i \leq 3$: For $\lambda = 1$, we find the eigenvector $\mathbf{v}_1 = (1, 5, 1)$ since elementary row operations give

$$A - I = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 3 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

For $\lambda = 2$, we find the eigenvector $\mathbf{v}_2 = (-3, -4, 1)$ since elementary row operations give

$$A - 2I = \begin{pmatrix} 2 & -1 & 2 \\ 1 & -1 & -1 \\ 2 & -1 & 2 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

For $\lambda = 6$, we find the eigenvector $\mathbf{v}_3 = (1, 0, 1)$ since elementary row operations give

$$A - 6I = \begin{pmatrix} -2 & -1 & 2 \\ 1 & -5 & -1 \\ 2 & -1 & -2 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & -5 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It follows that A is diagonalizable and that the general solution of the system of differential equations is

$$\mathbf{y} = C_1\mathbf{v}_1 e^{\lambda_1 t} + C_2\mathbf{v}_2 e^{\lambda_2 t} + C_3\mathbf{v}_3 e^{\lambda_3 t} = C_1 \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix} e^{2t} + C_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{6t}$$