EVALUATION GUIDELINES - Written examination

GRA 60353
Mathematics

Department of Economics

Start date:	08.01.2020	Time 13:00
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For more information about formalities, see examination paper.

Solutions	Final exam in GRA 6035 Mathematics
Date	January 8th, 2020 at 1300 - 1600

Question 1.

(a) We use Gaussian elimination to find the rank of A. We use a Gaussian process until we find an echelon form:

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 1 & 7 & -6 & 9 \\ 5 & 0 & 5 & 3 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -6 \\ 0 & 5 & -5 & 6 \\ 0 & -10 & 10 & -12 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -5 & 5 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are two pivot positions, we have that $\text{rk}(A) = 2$.

(b) Let us call the variables x, y, z, w. From the echelon form that we found in (a), we see that z and w are free, and back substitution gives that $-5y + 5z - 6w = 0$, or $-5y = -5z + 6w$, which gives $y = z - 6w/5$, and that $x + 2y - z + 3w = 0$, or $x = -2y + z - 3w$, which gives $x = -2(z - 6w/5) + z - 3w = -z - 3w/5$. Therefore, the solutions are given by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -z - 3w/5 \\ z - 6w/5 \\ z \\ w \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \frac{w}{5} \cdot \begin{pmatrix} -3 \\ -6 \\ 0 \\ 5 \end{pmatrix} = z \cdot \mathbf{w}_1 + \frac{w}{5} \cdot \mathbf{w}_2$$

It follows that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is a base of $\text{Null}(A)$, with

$$\mathbf{w}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} -3 \\ -6 \\ 0 \\ 5 \end{pmatrix}$$

(c) Since there are pivot positions in the first two columns of A, the first two column vectors of A is a base $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ of $\text{Col}(A)$. Since $\mathbf{w}_2$ is in $\text{Null}(A)$, we have that

$$A \cdot \mathbf{w}_2 = -3\mathbf{v}_1 - 6\mathbf{v}_2 + 5\mathbf{v}_4 = \mathbf{0} \quad \Rightarrow \quad 5\mathbf{v}_4 = 3\mathbf{v}_1 + 6\mathbf{v}_2$$

Question 2.

(a) We compute $\det(A)$ by cofactor expansion along the first column:

$$|A| = \begin{vmatrix} -7 & 6 & 2 \\ -6 & 5 & 2 \\ -6 & 6 & 1 \end{vmatrix} = -7(5-12) + 6(6-12) - 6(12-10) = 49 - 36 - 12 = 1$$

(b) We check if $\mathbf{v}_i$ is an eigenvector of A by computing $A\mathbf{v}_i$:

$$A\mathbf{v}_1 = \begin{pmatrix} -7 & 6 & 2 \\ -6 & 5 & 2 \\ -6 & 6 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot \mathbf{v}_1$$

$$A\mathbf{v}_2 = \begin{pmatrix} -7 & 6 & 2 \\ -6 & 5 & 2 \\ -6 & 6 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = -1 \cdot \mathbf{v}_2$$

$$A\mathbf{v}_3 = \begin{pmatrix} -7 & 6 & 2 \\ -6 & 5 & 2 \\ -6 & 6 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} = -1 \cdot \mathbf{v}_3$$

This means that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are eigenvectors of A, with eigenvalues $\lambda_1 = 1,\ \lambda_2 = \lambda_3 = -1$.

(c) It follows from the previous question that $\lambda_1 = 1$ is an eigenvalue of multiplicity one, and that $\lambda = -1$ is an eigenvalue of multiplicity 2. We have that E_1 has base $\{\mathbf{v}_1\}$, and since $\mathbf{v}_2$ and $\mathbf{v}_3$ are clearly linearly independent, E_{-1} has base $\{\mathbf{v}_2, \mathbf{v}_3\}$. Since there are enough eigenvalues and eigenvectors, A is diagonalizable.

Question 3.

(a) We solve the second order linear differential equation $y'' - 11y' + 18y = 9t^2 - 11t + 10$ using superposition. To find the homogeneous solution y_h, we consider the homogeneous differential equation $y'' - 11y' + 18y = 0$, which has characteristic equation $r^2 - 11r + 18 = 0$, with roots $r = 2$ and $r = 9$, and we have $y_h = C_1\, e^{2t} + C_2\, e^{9t}$. To find a particular solution y_p, we consider the differential equation $y'' - 11y' + 18y = 9t^2 - 11t + 10$ and use the method of undetermined coefficients. We start with $f(t) = 9t^2 - 11t + 10$, and compute $f'(t) = 18t - 11$ and $f''(t) = 18$. Based on this, we guess the solution $y = At^2 + Bt + C$, which gives $y' = 2At + B$ and $y'' = 2A$. When we substitute this into the differential equation, we get

$$(2A) - 11(2At + B) + 18(At^2 + Bt + C) = 9t^2 - 11t + 10$$

Collecting terms on the left-hand side, we get

$$18At^2 + (18B - 22A)t + (18C - 11B + 2A) = 9t^2 - 11t + 10$$

Comparing coefficients, we get $18A = 9$, or $A = 1/2$, $18B - 11 = -11$, or $B = 0$, and $18C + 1 = 10$, or $C = 1/2$. This means that $y_p = t^2/2 + 1/2$, and the general solution of the differential equation is therefore

$$y = y_h + y_p = C_1\, e^{2t} + C_2\, e^{9t} + \frac{1}{2}\, t^2 + \frac{1}{2}$$

(b) The differential equation $e^t\, y' = t\, y^2$ is separable, since it can be written in the form

$$y' = e^{-t} t y^2 = (te^{-t}) \cdot y^2 \quad \Rightarrow \quad \frac{1}{y^2} y' = te^{-t} \quad \Rightarrow \quad \int \frac{1}{y^2}\, \mathrm{d}y = \int te^{-t}\, \mathrm{d}t$$

The integral on the left-hand side can be solved by writing $1/y^2 = y^{-2}$, and we obtain

$$\int \frac{1}{y^2}\, \mathrm{d}y = \int y^{-2}\, \mathrm{d}y = -y^{-1} + C_1 = -\frac{1}{y} + C_1$$

The integral on the right-hand side can be solved using integration by parts, with $u' = e^{-t}$ and $v = t$, which gives $u = -e^{-t}$ and $v' = 1$, and therefore

$$\int te^{-t}\, \mathrm{d}t = uv - \int uv'\, \mathrm{d}t = -te^{-t} - \int (-e^{-t}) \cdot 1\, \mathrm{d}t = -te^{-t} - e^{-t} + C_2$$

The general solution can therefore be written as

$$-\frac{1}{y} + C_1 = -te^{-t} - e^{-t} + C_2 \quad \Rightarrow \quad \frac{1}{y} = te^{-t} + e^{-t} + C = \frac{t + 1 + Ce^t}{e^t}$$

in implicit form, with $C = C_1 - C_2$, and the general solution in explicit form is given by

$$y = \frac{e^t}{t + 1 + Ce^t}$$

(c) The system of differential equations has the form $\mathbf{y}' = A\mathbf{y}$ and we find the eigenvalues and eigenvectors of the matrix A: The characteristic equation is

$$\begin{vmatrix} -\lambda & 0 & 2 \\ 4 & -2-\lambda & 4 \\ 2 & 0 & -\lambda \end{vmatrix} = (-2-\lambda) \cdot (\lambda^2 - 4) = 0$$

by cofactor expansion along the middle column. This means that $\lambda_1 = \lambda_2 = -2$, $\lambda_3 = 2$. We need to compute a base for E_{-2} and E_2. We find the vectors $\mathbf{v}$ in the eigenspace E_{-2} by the Gaussian process

$$\begin{pmatrix} 2 & 0 & 2 \\ 4 & 0 & 4 \\ 2 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

and the vectors $\mathbf{v}$ in the eigenspace E_2 by the Gaussian process

$$\begin{pmatrix} -2 & 0 & 2 \\ 4 & -4 & 4 \\ 2 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} -2 & 0 & 2 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v} = z \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Since $\dim E_{-2} = 2$ and $\dim E_2 = 1$, A is diagonalizable, and $P^{-1}AP = D$ for the matrices D and P given

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

If we define new variables u_1, u_2, u_3 by $\mathbf{u} = P^{-1}\mathbf{y}$, which can also be written $\mathbf{y} = P\mathbf{u}$, then it follows that

$$\mathbf{u}' = (P^{-1}\mathbf{y})' = P^{-1}\mathbf{y}' = P^{-1}A\mathbf{y} = P^{-1}AP \cdot P^{-1}\mathbf{y} = D \cdot \mathbf{u}$$

Hence $u_i' = \lambda_i u_i$, which gives $u_i = C_i \cdot e^{\lambda_i t}$ for $1 \le i \le 3$, and

$$\mathbf{y} = P\mathbf{u} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} C_1 e^{-2t} \\ C_2 e^{-2t} \\ C_3 e^{2t} \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-2t} + C_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{2t}$$

Question 4.

(a) Since f is a quadratic form, we may either use the symmetric matrix A of f, or the Hessian matrix $H(f) = 2A$ of f. We choose to use the symmetric matrix A, which is given by

$$A = \begin{pmatrix} -4 & 0 & 2 & 2 \\ 0 & -10 & -2 & 2 \\ 2 & -2 & -5 & 3 \\ 2 & 2 & 3 & -5 \end{pmatrix}$$

We compute its first leading principal minors $D_1 = -4 < 0$, $D_2 = 40 > 0$, and

$$D_3 = -4(50-4) + 2(0+20) = -144 < 0$$

Moreover, we have that $D_4 = |A| = 0$ since $H(f) = 2A$ and $\det H(f) = 2^4 \cdot |A| = 0$. It follows that $\operatorname{rk} A = 3$, and A is negative semidefinite by the reduced rank criterion (RRC). Therefore $H(f) = 2A$ is also negative semidefinite, and f is a concave function.

(b) The Lagrangian is $\mathcal{L} = f(x,y,z,w) - \lambda(x^2 + y^2 + z^2 + w^2)$, and the first order conditions (FOC) are given by

$$\mathcal{L}'_x = -8x + 4z + 4w - \lambda \cdot 2x = 0$$
$$\mathcal{L}'_y = -20y - 4z + 4w - \lambda \cdot 2y = 0$$
$$\mathcal{L}'_z = 4x - 4y - 10z + 6w - \lambda \cdot 2z = 0$$
$$\mathcal{L}'_w = 4x + 4y + 6z - 10w - \lambda \cdot 2w = 0$$

and the constraint (C) is given by $x^2 + y^2 + z^2 + w^2 = 6$. We see that when $\lambda = -12$, the first order conditions is a linear system, with coefficient matrix

$$\begin{pmatrix} 16 & 0 & 4 & 4 \\ 0 & 4 & -4 & 4 \\ 4 & -4 & 14 & 6 \\ 4 & 4 & 6 & 14 \end{pmatrix}$$

We solve the linear system using Gaussian elimination, and start by dividing the first row with 4. We obtain the following echelon form:

$$\begin{pmatrix} 4 & 0 & 1 & 1 \\ 0 & 4 & -4 & 4 \\ 4 & -4 & 14 & 6 \\ 4 & 4 & 6 & 14 \end{pmatrix} \to \begin{pmatrix} 4 & 0 & 1 & 1 \\ 0 & 4 & -4 & 4 \\ 0 & -4 & 13 & 5 \\ 0 & 4 & 5 & 13 \end{pmatrix} \to \begin{pmatrix} 4 & 0 & 1 & 1 \\ 0 & 4 & -4 & 4 \\ 0 & 0 & 9 & 9 \\ 0 & 0 & 9 & 9 \end{pmatrix} \to \begin{pmatrix} 4 & 0 & 1 & 1 \\ 0 & 4 & -4 & 4 \\ 0 & 0 & 9 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that w is free, and use back substitution to solve the linear system. We have that $9z + 9w = 0$, or $z = -w$, that $4y - 4z + 4w = 0$, or $y = z - w = -2w$, and that $4x + z + w = 0$, or $4x = -z - w = 0$, which gives $x = 0$. This implies that $(x, y, z, w; \lambda) = (0, -2w, -w, w; -12)$ with w free. We put this into the constraint, and get

$$x^2 + y^2 + z^2 + w^2 = 0^2 + (-2w)^2 + (-w)^2 + w^2 = 3 \quad \Rightarrow \quad 6w^2 = 6$$

This means that $w^2 = 1$, or $w = \pm 1$. We find the following solutions of the Lagrange conditions with $\lambda = -12$:

$$(x, y, z, w; \lambda) = (0, -2, -1, 1; -12), (0, 2, 1, -1; -12)$$

(c) We consider the function $h(x, y, z, w) = \mathcal{L}(x, y, z, w; -12)$ and try to use the SOC to determine whether the candidate points we found in (b) are minimum points. The function h is given by

$$h(x, y, z, w) = 8x^2 + 2y^2 + 7z^2 + 7w^2 + 4xz + 4xw - 4yz + 4yw + 6zw$$

The Hessian matrix of h is given by

$$H(h) = \begin{pmatrix} 16 & 0 & 4 & 4 \\ 0 & 4 & -4 & 4 \\ 4 & -4 & 14 & 6 \\ 4 & 4 & 6 & 14 \end{pmatrix}$$

We have $D_1 = 16$, $D_2 = 64$ and $D_3 = 16(56-16)+4(0-16) = 576$. Moreover, $D_4 = |H(h)| = 0$ since $H(h)$ is the matrix from (b), with one free variable. Hence, it follows from the reduced rank condition (RRC) that $H(h)$ is positive semidefinite, and therefore h is convex. By the SOC, this means that the candidate points $(x, y, z, w; \lambda) = (0, -2, -1, 1; -12), (0, 2, 1, -1; -12)$ are minimum points. The minimum value is $f(0, 2, 1, -1) = -72$.

(d) Any candidate point $(x, y, z, w; \lambda)$ that satisfies FOC+C with $\lambda = 0$ must be a maximum point. In fact, $h(x, y, z, w) = \mathcal{L}(x, y, z, w; 0) = f(x, y, z, w)$ is concave, and therefore this follows from the SOC. We try to find such candidate points, and use the FOC from (b) with $\lambda = 0$ instead of $\lambda = -12$. The linear system we get has coefficient matrix

$$\begin{pmatrix} -8 & 0 & 4 & 4 \\ 0 & -20 & -4 & 4 \\ 4 & -4 & -10 & 6 \\ 4 & 4 & 6 & -10 \end{pmatrix}$$

This matrix is $2A$, where A is the symmetric matrix of f, so we know that there will be at least one free variable since $|A| = 0$. We use Gaussian elimination, and find an echelon form:

$$\begin{pmatrix} -8 & 0 & 4 & 4 \\ 0 & -20 & -4 & 4 \\ 4 & -4 & -10 & 6 \\ 4 & 4 & 6 & -10 \end{pmatrix} \to \begin{pmatrix} -8 & 0 & 4 & 4 \\ 0 & -20 & -4 & 4 \\ 0 & -4 & -8 & 8 \\ 0 & 4 & 8 & -8 \end{pmatrix} \to \begin{pmatrix} -8 & 0 & 4 & 4 \\ 0 & -4 & -8 & 8 \\ 0 & 0 & 36 & -36 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We switched the two middle rows to make the last operations easier. From this echelon form, we see that w is free, that $z = w$, that $-4y = 8z - 8w = 0$, and that $-8x = -4z - 4w = -8w$, or $x = w$. It follows that $(x, y, z, w) = (w, 0, w, w)$, and the constraint gives

$$x^2 + y^2 + z^2 + w^2 = 6 \quad \Rightarrow \quad w^2 + 0^2 + w^2 + w^2 = 3w^2 = 6$$

This gives $w^2 = 2$, or $w = \pm\sqrt{2}$. It follows that there are two candidate points

$$(x, y, z, w; \lambda) = (\sqrt{2}, 0, \sqrt{2}, \sqrt{2}; 0), (-\sqrt{2}, 0, -\sqrt{2}, -\sqrt{2}; 0)$$

that satisfy FOC+C. By the comments above, these points are maximum points, with maximum value $f(\sqrt{2}, 0, \sqrt{2}, \sqrt{2}) = 0$.