EVALUATION GUIDELINES - Written examination

GRA 60353
Mathematics

Department of Economics

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For more information about formalities, see examination paper.

Solutions	Final exam in GRA 6035 Mathematics
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Question 1.

(a) We use Gaussian elimination to find the rank of A. We start by switching the first two rows and obtain zeros under the first pivot:

$$A = \begin{pmatrix} 2 & 1 & 5 & 9 \\ -1 & 1 & 2 & -3 \\ 3 & 0 & 1 & 10 \\ 0 & 3 & 0 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 2 & -3 \\ 2 & 1 & 5 & 9 \\ 3 & 0 & 1 & 10 \\ 0 & 3 & 0 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 2 & -3 \\ 0 & 3 & 9 & 3 \\ 0 & 3 & 7 & 1 \\ 0 & 3 & 0 & -6 \end{pmatrix}$$

Then we obtain zeros under the next pivots:

$$\begin{pmatrix} -1 & 1 & 2 & -3 \\ 0 & 3 & 9 & 3 \\ 0 & 3 & 7 & 1 \\ 0 & 3 & 0 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 2 & -3 \\ 0 & 3 & 9 & 3 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -9 & -9 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 2 & -3 \\ 0 & 3 & 9 & 3 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are three pivot positions, we have that $\text{rk}(A) = 3$.

(b) We have that $\dim \text{Null}(A) = n - \text{rk}(A) = 4 - 3 = 1$ since the dimension equals the number of degrees of freedom. Let us call the variables x, y, z, w. From the echelon form that we found in (a), we see that w is free, and back substitution gives that $-2z - 2w = 0$, or $z = -w$, that

$$3y + 9z + 3w = 3y + 9(-w) + 3w = 0 \quad \Rightarrow \quad 3y = 6w \quad \Rightarrow \quad y = 2w$$

and that

$$-x + y + 2z - 3w = -x + 2w + 2(-w) - 3w = 0 \quad \Rightarrow \quad -x = 3w \quad \Rightarrow \quad x = -3w$$

Therefore, the solutions are given by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -3w \\ 2w \\ -w \\ w \end{pmatrix} = w \cdot \begin{pmatrix} -3 \\ 2 \\ -1 \\ 1 \end{pmatrix} = w \cdot \mathbf{v} \quad \text{with} \quad \mathbf{v} = \begin{pmatrix} -3 \\ 2 \\ -1 \\ 1 \end{pmatrix}$$

and therefore $\mathcal{B} = \{\mathbf{v}\}$ is a base of $\text{Null}(A)$.

(c) Since $\mathbf{v}$ is in $\text{Null}(A)$, we have that $A \cdot \mathbf{v} = \mathbf{0}$, and it follows that

$$A \cdot \mathbf{v} = -3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0} \quad \Rightarrow \quad \mathbf{v}_4 = 3\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$$

Question 2.

(a) We check if $\mathbf{v}$ is an eigenvector of A by computing $A\mathbf{v}$:

$$A\mathbf{v} = \begin{pmatrix} 4 & 0 & 6 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since $A\mathbf{v} = \mathbf{0}$, we have that $A\mathbf{v} = \lambda\mathbf{v}$ with $\lambda = 0$, and $\mathbf{v}$ is an eigenvector of A with eigenvalue $\lambda = 0$.

(b) We solve the characteristic equation to find the eigenvalues of A, given by

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & 6 \\ -1 & 3-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

We use cofactor expansion along the first row to compute the determinant, and get

$$(4-\lambda)((3-\lambda)(2-\lambda) - 0) + 6(-1 - (3-\lambda)) = (4-\lambda)(\lambda^2 - 5\lambda + 6) + 6(\lambda - 4)$$

We see that $\lambda - 4$ is a common factor, and write the characteristic equation in factorized form

$$(4-\lambda)(\lambda^2 - 5\lambda + 6 - 6) = 0$$

This gives $\lambda = 4$, or $\lambda^2 - 5\lambda = 0$, which gives $\lambda = 0$ or $\lambda = 5$. We conclude that the eigenvalues of A are $\lambda = 0, \ \lambda = 4, \ \lambda = 5$.

(c) Since A is a 3×3 matrix with three distinct eigenvalues, if follows that A is diagonalizable. In fact, the eigenspaces E_0, E_4 and E_5 all have dimension one, and therefore there are eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in the three eigenspaces such that $P^{-1}AP$ is diagonal when $P = (\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)$. In fact, we can use $\mathbf{v}_1 = \mathbf{v}$ as the first eigenvector.

Question 3.

(a) We solve the first order linear differential equation $y' - 4y = 10\,e^{-t}$ using superposition. To find the homogeneous solution y_h, we consider the homogeneous differential equation $y' - 4y = 0$, which has characteristic equation $r - 4 = 0$, with root $r = 4$, and we have $y_h = C\,e^{4t}$. To find the particular solution y_p, we consider the differential equation $y' - 4y = 10\,e^{-t}$ and use the method of undetermined coefficients. We start with $f(t) = 10\,e^{-t}$, and compute $f'(t) = -10\,e^{-t}$. Based on this, we guess the solution $y = A\,e^{-t}$, which gives $y' = -A\,e^{-t}$. When we substitute this into the differential equation, we get

$$(-A\,e^{-t}) - 4(A\,e^{-t}) = 10\,e^{-t} \quad \Rightarrow \quad -5A\,e^{-t} = 10\,e^{-t}$$

Comparing coefficients, we get $-5A = 10$, or $A = -2$, and $y_p = -2\,e^{-t}$. The general solution of the differential equation is therefore

$$y = y_h + y_p = C\,e^{4t} - 2e^{-t}$$

Alternatively, it is possible to solve the differential equation using integrating factor.

(b) We try to solve the differential equation $2t + 2ty^2 + (2y + 2yt^2)y' = 0$ as an exact differential equation, and look for a function $h = h(t, y)$ such that

$$h'_t = 2t + 2ty^2, \quad h'_y = 2y + 2yt^2$$

From the first condition, we get that $h = t^2 + t^2y^2 + \phi(y)$, and when we substitute this into the second condition, we get

$$h'_y = (t^2 + t^2y^2 + \phi(y))'_y = 0 + t^2 \cdot 2y + \phi'(y) = 2yt^2 + \phi'(y) = 2y + 2yt^2$$

We see that this is satisfied if $\phi'(y) = 2y$, and we may choose $\phi(y) = y^2$. Therefore, the diffential equation is in exact form $h'_t + h'_y\,y' = 0$ for $h(t, y) = t^2 + t^2y^2 + y^2$, and the general solution is given by

$$h(t, y) = t^2 + t^2y^2 + y^2 = C \quad \Rightarrow \quad y^2(1 + t^2) = C - t^2 \quad \Rightarrow \quad y = \pm\sqrt{\frac{C - t^2}{1 + t^2}}$$

Alternatively, it is possible to solve the differential equation as a separable differential equation, since it can be written in the form

$$2y(t^2 + 1)y' = -2t(1 + y^2) \quad \Rightarrow \quad y' = \frac{-2t(1 + y^2)}{2y(t^2 + 1)} = \frac{-2t}{t^2 + 1} \cdot \frac{y^2 + 1}{2y}$$

(c) The system of differential equations can be written in the form $\mathbf{y}' = A\mathbf{y}$, where A is the matrix in Question 2. We found that A is diagonalizable in 2 (c), with eigenvalues $\lambda = 0$, $\lambda = 4$ and $\lambda = 5$. The vector $\mathbf{v}$ from Question 2 can be used as a base for E_0, and we need to compute a base for E_4 and E_5. We find the vectors $\mathbf{v}$ in the eigenspace E_4 by the Gaussian process

$$\begin{pmatrix} 4-4 & 0 & 6 \\ -1 & 3-4 & 0 \\ 1 & 1 & 2-4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ -1 & -1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v} = y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

and the vectors $\mathbf{v}$ in the eigenspace E_5 by the Gaussian process

$$\begin{pmatrix} 4-5 & 0 & 6 \\ -1 & 3-5 & 0 \\ 1 & 1 & 2-5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -3 \\ -1 & -2 & 0 \\ -1 & 0 & 6 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 6 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v} = z \cdot \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$$

This means that $P^{-1}AP = D$ for the matrices D and P given by eigenvalues and eigenvectors:

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad P = \begin{pmatrix} -3 & -1 & 6 \\ -1 & 1 & -3 \\ 2 & 0 & 1 \end{pmatrix}$$

If we define new variables u_1, u_2, u_3 by $\mathbf{u} = P^{-1}\mathbf{y}$, which can also be written $\mathbf{y} = P\mathbf{u}$, then it follows that

$$\mathbf{u}' = (P^{-1}\mathbf{y})' = P^{-1}\mathbf{y}' = P^{-1}A\mathbf{y} = P^{-1}AP \cdot P^{-1}\mathbf{y} = D \cdot \mathbf{u}$$

Hence $u_i' = \lambda_i u_i$, which gives $u_i = C_i \cdot e^{\lambda_i t}$ for $1 \le i \le 3$, and

$$\mathbf{y} = P\mathbf{u} = \begin{pmatrix} -3 & -1 & 6 \\ -1 & 1 & -3 \\ 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} C_1\, e^0 \\ C_2\, e^{4t} \\ C_3\, e^{5t} \end{pmatrix} = C_1 \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{4t} + C_3 \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} e^{5t}$$

Question 4.

(a) The Hessian matrix of f is given by $H(f) = 2A$, where A is the symmetric matrix of the quadratic form, or

$$H(f) = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

We have leading principal minors $D_1 = 2$, $D_2 = 4 - 1 = 3$, and $D_3 = |A|$ is given by cofactor expansion along the first row:

$$D_3 = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 2(4-1) - (-1)(-2+1) + 1(1-2) = 4$$

Since all leading principal minors are positive, $H(f)$ is positive definite, and f is a convex function.

(b) The Lagrangian of the Lagrange problem is $\mathcal{L} = x^2 + y^2 + z^2 - xy + xz - yz - \lambda(x + y + z)$, and the first order conditions (FOC) are

$$\begin{aligned} \mathcal{L}_x' &= 2x - y + z - \lambda = 0 \\ \mathcal{L}_y' &= -x + 2y - z - \lambda = 0 \\ \mathcal{L}_z' &= x - y + 2z - \lambda = 0 \end{aligned}$$

and the constraint (C) is given by $x + y + z = 11$. We see that the Lagrange conditions is a linear system, with augmented matrix

$$\left(\begin{array}{rrrr|r} 2 & -1 & 1 & -1 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 0 & 11 \end{array}\right)$$

We solve the linear system using Gaussian elimination, and start by switching the second row to the first row:

$$\left(\begin{array}{rrrr|r} 1 & 1 & 0 & -2 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 0 & 11 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 0 & -2 & 0 \\ 0 & 3 & -1 & -3 & 0 \\ 0 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 11 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & -2 & 2 & 1 & 0 \end{array}\right)$$

In the last step, we added the third row to the second row, and switched the third and fourth row. Then we continue until we get an ecehelon form:

$$\left(\begin{array}{rrrr|r} 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & 4 & -3 & 0 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & -11 & -44 \end{array}\right)$$

From the echelon form, we use back substitution to solve the linear system, and find that $\lambda = 4$, that $z = 11 - 2 \cdot 4 = 3$, that $y = -3 + 2 \cdot 4 = 5$, and that $x = -5 + 2 \cdot 4 = 3$. From this computation, it follows that $(x, y, z; \lambda) = (3, 5, 3; 4)$ is the unique solution of the Lagrange conditions FOC+C. We use the SOC, and see that

$$h(x, y, z) = \mathcal{L}(x, y, z; 4) = f(x, y, z) - 4(x + y + z)$$

has the same Hessian matrix as f. Since f is convex from (a), the same applies to h, and $f_{min} = f(3,5,3) = 22$ by the SOC.

(c) By the envelope theorem, the optimal value function $f^*(a)$ of the Lagrange problem

$$\min\ f(x,y,z) = x^2 + y^2 + z^2 - xy + xz - yz \text{ subject to } x + y + z = a$$

has derivative $df^*(a)/da = \lambda^*(a)$, and $\lambda^*(11) = 4$ at $a = 11$ by the computation in (b). Hence, we estimate that the minimum value

$$f^*(10) \approx f^*(11) + (10 - 11) \cdot 4 = 22 - 4 = 18$$

Question 5.

Using eigenvalues and eigenvectors from Question 3 (c), we have that

$$\mathbf{y}_t = C_1 \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} 0^t + C_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} 4^t + C_3 \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} 5^t$$

This means that for $t \geq 1$, we get the general solution

$$\mathbf{y}_t = C_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} 4^t + C_3 \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} 5^t$$

For $t = 0$, we have that

$$\mathbf{y}_0 = C_1 \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$$

We solve the equation given by the initial condition using Gaussian elimination:

$$\left(\begin{array}{ccc|c} -3 & -1 & 6 & 1 \\ -1 & 1 & -3 & 1 \\ 2 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 1 & -3 & 1 \\ 0 & -4 & -15 & -2 \\ 0 & 2 & -5 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 1 & -3 & 1 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 5 & 4 \end{array}\right)$$

This gives $C_3 = 4/5$, $C_2 = 7/2$, and $C_1 = 1/10$ by back substitution, and the particular solution is

$$\mathbf{y}_t = \frac{7}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} 4^t + \frac{4}{5} \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} 5^t \quad \text{for } t \geq 1, \quad \mathbf{y}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$