Solutions	Final exam in GRA 6035 Mathematics
Date	January 7th, 2019 at 1300 - 1600

Question 1.

(a) We compute the leading principal minors of A, and find that $D_1 = 2$, $D_2 = 4 - 1 = 3$, $D_3 = 3D_2 = 9$, and

$$D_4 = 2 \cdot 2 \cdot (-9 - 16) - 1 \cdot 1 \cdot (-9 - 16) = (4 - 1)(-25) = -75$$

Since $D_4 < 0$, we conclude that A is indefinite. Alternatively, it is possible to see this from the fact A has both positive and negative principal minors of order one, since $\Delta_1 = 2, 2, 3, -3$.

(b) The eigenvectors of A with eigenvalue $\lambda = -5$ are the solutions of $(A + 5I) \cdot \mathbf{x} = \mathbf{0}$. We use Gaussian elimination to find these solutions:

$$\left(\begin{array}{cccc|c} 7 & 1 & 0 & 0 & 0 \\ 1 & 7 & 0 & 0 & 0 \\ 0 & 0 & 8 & 4 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 7 & 0 & 0 & 0 \\ 7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 8 & 4 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 7 & 0 & 0 & 0 \\ 0 & -48 & 0 & 0 & 0 \\ 0 & 0 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We see that there is one free variable, so $\dim E_{-5} = 1$. Backward substitution gives $8z+4w = 0$, or $z = -w/2$, that $-48y = 0$, or $y = 0$, and that $x+7y = 0$, or $x = -7y = 0$. The eigenvectors are therefore given by

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ -w/2 \\ w \end{pmatrix} = w \cdot \begin{pmatrix} 0 \\ 0 \\ -1/2 \\ 1 \end{pmatrix} = w \cdot \mathbf{v} \quad \text{with } \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ -1/2 \\ 1 \end{pmatrix}$$

The eigenvectors can therefore be written as $E_{-5} = \text{span}(v)$.

(c) By definition, $\dim \text{Null}(A - rI) \geq 1$ if and only if $\det(A - rI) = 0$, or if r is an eigenvalue for A. We compute the eigenvalues:

$$\det(A - rI) = \begin{vmatrix} 2-r & 1 & 0 & 0 \\ 1 & 2-r & 0 & 0 \\ 0 & 0 & 3-r & 4 \\ 0 & 0 & 4 & -3-r \end{vmatrix} = 0$$

To simplify the computation, we first compute the 2-minor in the lower right corner, which is $(3 - r)(-3 - r) - 16 = r^2 - 9 - 16 = r^2 - 25$. We then compute the full determinant using cofactor expansion along the first row:

$$(2 - r)(2 - r) \cdot (r^2 - 25) - 1 \cdot 1 \cdot (r^2 - 25) = ((2 - r)^2 - 1) \cdot (r^2 - 25) = 0$$

We end up with the equation $(r^2 - 4r + 3)(r^2 - 25) = 0$, with solutions $r = 1$, $r = 3$, $r = 5$ and $r = -5$. Hence A has eigenvalues $r = 1, 3, 5, -5$.

Question 2.

(a) The differential equation $4y'' - 4y' - 3y = 9t$ is second order linear and we can solve it using superposition. To find the homogeneous solution y_h, we consider the homogeneous differential equation $4y'' - 4y' - 3y = 0$, which has characteristic equation $4r^2 - 4r - 3 = 0$, with two distinct solutions $r = 3/2$ and $r = -1/2$. Therefore, we have

$$y_h = C_1\, e^{3t/2} + C_2\, e^{-t/2}$$

To find the particular solution y_p, we consider the differential equation $4y'' - 4y' - 3y = 9t$ and use the method of undetermined coefficients. We start with $f(t) = 9t$, and compute $f' = 9$ and $f'' = 0$. Based on this, we guess the solution $y = At + B$, which gives $y' = A$ and $y'' = 0$. When we substitute this into the differential equation, we get

$$-4A - 3(At + B) = 9t$$

Comparing coefficients, we get $-3A = 9$ and $-4A - 3B = 0$, or $A = -3$ and $B = 4$, and

$$y_p = -3t + 4$$

The general solution of the differential equation is therefore

$$y = y_h + y_p = C_1\, e^{3t/2} + C_2\, e^{-t/2} - 3t + 4$$

(b) The differential equation $4ty' + 4y = 1$ can be written $4y - 1 + 4ty' = 0$, and we try to solve it as an exact differential equation, and look for a function $h(t, y)$ such that

$$h'_t = 4y - 1, \quad h'_y = 4t$$

We see that $h = 4ty - t$ is one solution, so the differential equation is exact and the general solutions is given by

$$4ty - t = C \quad \Rightarrow \quad y = \frac{C + t}{4t}$$

(c) The system of differential equations can be written in the form $\mathbf{y}' = A \cdot \mathbf{y}$, where

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

We find the eigenvalues and eigenvectors of A: The eigenvalues of A are the diagonal entries $\lambda_1 = 1$ and $\lambda_2 = 5$, since A is upper triangular. Since all eigenvalues have multiplicity one, the eigenspaces are one-dimensinal and can be written $E_1 = \text{span}(\mathbf{v}_1)$ and $E_5 = \text{span}(\mathbf{v}_2)$. In fact, we may choose

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

since we have that

$$A - \lambda_1 I = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}, \quad A - \lambda_2 I = \begin{pmatrix} -4 & 2 \\ 0 & 0 \end{pmatrix}$$

Since A is diagonalizable, with enough eigenvalues and eigenvectors, we have that the general solution is

$$\mathbf{y} = C_1 \mathbf{v}_1\, e^{\lambda_1 t} + C_2 \mathbf{v}_2\, e^{\lambda_2 t} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} = \begin{pmatrix} C_1\, e^t & + & C_2\, e^{5t} \\ & & 2C_2\, e^{5t} \end{pmatrix}$$

Alternatively, we could solve the second equation $y_2' = 5y_2$ as a linear differential equation, which gives $y_2 = C_2\, e^{5t}$, and then substitute this in the first differential equation, and get

$$y_1' = y_1 + 2y_2 = y_1 + 2C_2\, e^{5t} \quad \Rightarrow \quad y_1' - y_1 = 2C_2\, e^{5t}$$

We can solve this as a linear differential equation, and get $y_1 = y_1^h + y_1^p = C_1\, e^t + \frac{1}{2} C_2\, e^{5t}$.

Question 3.

(a) The first order partial derivatives of $f(x, y, z) = 3x^2 + y^2 + axy - y + 2z^4 + 8z + 12$ and the FOC's are given by

$$f'_x = 6x + ay = 0, \quad f'_y = 2y + ax - 1 = 0, \quad f'_z = 8z^3 + 8 = 0$$

When $a = 3$, the last FOC gives $8z^3 = -8$, or $z = -1$, and the first FOC gives $y = -2x$. Substituting this in the middle FOC, we get $-4x + 3x - 1 = 0$, or $x = -1$, and this gives $y = -2x = 2$. There is a unique stationary point when $a = 3$, given by $(x, y, z) = (-1, 2, -1)$.

(b) The Hessian matrix of f is given by

$$H(f) = \begin{pmatrix} 6 & a & 0 \\ a & 2 & 0 \\ 0 & 0 & 24z^2 \end{pmatrix}$$

For all x, y, z, we have that $D_1 = 6 > 0$, that $D_2 = 12 - a^2$, and that $D_3 = 24z^2(12 - a^2)$. We know that f is convex if and only if $H(f)$ is positive semi-definite for all x, y, z. So if f is convex, then $D_2 = 12 - a^2 \geq 0$. Conversely, if $12 - a^2 \geq 0$, then $D_1, D_2, D_3 \geq 0$, and all principal minors $\Delta_1 = 6, 2, 24z^2 \geq 0$, $\Delta_2 = 12 - a^2, 144z^2, 48z^2 \geq 0$, and $\Delta_3 = 24z^2(12 - a^2) \geq 0$, which means that f is convex. It follows that f is convex if and only if $a^2 \leq 12$, or $-\sqrt{12} \leq a \leq \sqrt{12}$.

(c) When $a = 3$, it follows from (b) that f is convex, and the stationary point $(-1, 2, -1)$ found in (a) is a global minimum point, with minimum value $f^*(3) = f(-1, 2, -1) = 5$. By the envelope theorem, we have that

$$\frac{df^*(a)}{da} = f'_a(x^*(a), y^*(a), z^*(a))$$

Since $f'_a = xy$, it follows that $df^*(a)/da = x^*(a) \cdot y^*(a)$, and $df^*(a)/da = (-1) \cdot 2 = -2$ at $a = 3$. This means that the minimum value is

$$f^*(a) \approx f^*(3) + \Delta a \cdot \frac{df^*(a)}{da} = 5 + (a-3)(-2) = 11 - 2a$$

when a is close to 3. Note that f is convex when a is close to 3 by (b), and there are stationary points of f such that a global minimum exists.

Question 4.

(a) The Lagrangian is $\mathcal{L} = 2x^2 + 2xy + 2y^2 + 3z^2 + 8zw - 3w^2 - \lambda(x^2 + y^2 + z^2 + w^2)$. The first order conditions (FOC) are

$$\begin{aligned} \mathcal{L}'_x &= 4x + 2y - \lambda(2x) = 0 \\ \mathcal{L}'_y &= 2x + 4y - \lambda(2y) = 0 \\ \mathcal{L}'_z &= 6z + 8w - \lambda(2z) = 0 \\ \mathcal{L}'_w &= 8z - 6w - \lambda(2w) = 0 \end{aligned}$$

and the constraint (C) is given by $x^2 + y^2 + z^2 + w^2 = 1$. The Lagrange conditions are FOC+C.

(b) When $\lambda = -5$, the first order conditions become

$$\begin{aligned} \mathcal{L}'_x &= 4x + 2y + 10x = 0 \\ \mathcal{L}'_y &= 2x + 4y + 10y = 0 \\ \mathcal{L}'_z &= 6z + 8w + 10z = 0 \\ \mathcal{L}'_w &= 8z - 6w + 10w = 0 \end{aligned}$$

The two first FOC's give $14x + 2y = 0$, or $y = -7x$, and $2x + 14y = 2x + 14(-7x) = 0$, or $-96x = 0$. This gives $x = y = 0$. The last two FOC's give $16z + 8w = 0$, or $w = -2z$, and $8z + 4w = 8z + 4(-2z) = 0$, or $0z = 0$. This gives $z = -w/2$ and w free, and the solutions of the FOC's are therefore given by $(x, y, z, w) = (0, 0, -w/2, w)$ with w free. Alternatively, we could find these solutions using Gaussian elimnation, as the FOC's are linear. Finally, the constraint gives $(-w/2)^2 + w^2 = 1$, or $5w^2/4 = 1$. This implies that $w^2 = 4/5$, or $w = \pm 2/\sqrt{5}$. The points with $\lambda = -5$ that satisfy the Lagrange conditions are therefore

$$(x, y, z, w; \lambda) = (0, 0, -1/\sqrt{5}, 2/\sqrt{5}; -5), (0, 0, 1/\sqrt{5}, -2/\sqrt{5}; -5)$$

(c) We apply the SOC (second order condition) to the candidate points $(0, 0, \mp - 1/\sqrt{5}, \pm 2/\sqrt{5})$ with $\lambda = -5$: By the SOC, these points are solutions of the Lagrange problem if the function

$$\begin{aligned} h(x, y, z, w) &= \mathcal{L}(x, y, z, w; -5) \\ &= 2x^2 + 2xy + 2y^2 + 3z^2 + 8zw - 3w^2 + 5(x^2 + y^2 + z^2 + w^2) \\ &= 7x^2 + 2xy + 7y^2 + 8z^2 + 8zw + 2w^2 \end{aligned}$$

is convex. The Hessian matrix of h is

$$H(h) = \begin{pmatrix} 14 & 2 & 0 & 0 \\ 2 & 14 & 0 & 0 \\ 0 & 0 & 16 & 8 \\ 0 & 0 & 8 & 4 \end{pmatrix}$$

We compute its leading principal minors, which are $D_1 = 14 > 0$, $D_2 = 196 - 4 = 192 > 0$, $D_3 = 16 D_2 > 0$ and

$$D_4 = 14 \cdot 14 \cdot (16 \cdot 4 - 8^2) - 2 \cdot 2 \cdot (16 \cdot 4 - 8^2) = 0$$

Alternatively, we could see that $D_4 = 0$ from the fact that the last row in $H(h)$ is $1/2$ times the third row. Since $D_1, D_2, D_3 > 0$ and $D_4 = 0$, it follows that $\operatorname{rk} H(h) = 3$ and $H(h)$ is positive semi-definite by the RRC (reduced rank condition). Hence h is convex, and by the SOC, the candidates found in (b) solves the Lagrange problem, with $f^*_{\text{min}} = f(0, 0, -1/\sqrt{5}, 2/\sqrt{5}) = -5$.

Question 5.

The objective function is the quadratic form $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ with symmetric matrix A, where A is the matrix from Question 1, and the constraint can be written as $g(x, y, z, w) \leq 1$, where g is the quadratic form $g(\mathbf{x}) = \mathbf{x}^T I\mathbf{x}$ with symmetric matrix I, the identity matrix. This means that the Lagrangian of the Kuhn-Tucker problem is the quadratic form given by

$$\mathcal{L}(x, y, z, w; \lambda) = \mathbf{x}^T A\mathbf{x} - \lambda(\mathbf{x}^T I\mathbf{x}) = \mathbf{x}^T (A - \lambda I)\mathbf{x}$$

It follows that the first order conditions is a linear system, and it can be written as

$$2(A - \lambda I)\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad (A - \lambda I)\mathbf{x} = \mathbf{0}$$

Therefore, we have that $(\mathbf{x}; \lambda)$ is a solution of the FOC's if and only if $\mathbf{x}$ is an eigenvector of A with eigenvalue λ. If $\lambda = 0$, then FOC's implies that $\mathbf{x} = \mathbf{0}$ since $\lambda = 0$ is not an eigenvalue; the eigenvalues of A are $\lambda = 1, 3, 5, -5$ from Question 1(c). We therefore get the candidate point $(0, 0, 0, 0; 0)$ with $f = 0$ from the non-binding case. In the binding case $g(\mathbf{x}) = 1$, any solution of the FOC's is an eigenvector with eigenvalue $\lambda \neq 0$. For each possible eigenvalue $\lambda > 0$, there are two points that satisfy the constraint: In fact, $E_\lambda = \operatorname{span}(\mathbf{v})$ since all eigenvalues have multiplicty one, and $\mathbf{x} = c\mathbf{v}$ gives

$$g(\mathbf{x}) = \mathbf{x}^T\mathbf{x} = (c\mathbf{v})^T(c\mathbf{v}) = c^2\mathbf{v}^T\mathbf{v} = 1 \quad \Rightarrow \quad c = \pm\sqrt{\frac{1}{\mathbf{v}^T\mathbf{v}}}$$

which gives two vectors since $\mathbf{v}^T\mathbf{v} > 0$. For each of these two vectors in E_λ that satisfy the constraint, we have

$$f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T \lambda\mathbf{x} = \lambda\mathbf{x}^T I\mathbf{x} = \lambda \cdot 1 = \lambda$$

The best candidate for maximum is therefore one of the two eigenvectors with maximal eigenvalue $\lambda = 5$ that satisfies the constraint. It is clear that the constraint give a bounded set of admissible points, so the problem has a maximum by the EVT (extreme value theorem). Finally, the NDCQ is satisfied for all admissible points. We can therefore conclude that $f^*_{\text{max}} = 5$ is the maximum value of the Kuhn-Tucker problem.