Exam	Final exam in GRA 6035 Mathematics
Date	November 27th, 2018 at 0900 - 1200

This exam consists of 12+1 problems (one additional problem is for extra credits, and can be skipped). Each problem has a maximal score of 6p, and 72p (12 solved problems) is marked as 100% score.

You must give reasons for your answers. Precision and clarity will be emphasized when evaluating your answers.

Question 1.

We consider the matrix A given by

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 8 \\ -1 & 0 & 1 & 0 \\ 0 & 8 & 0 & -2 \end{pmatrix}$$

(a) **(6p)** Compute the rank of A. How many free variables does $A \cdot \mathbf{x} = \mathbf{0}$ have?
(b) **(6p)** Find Null(A), and determine its dimension.
(c) **(6p)** Determine the definiteness of A.

Question 2.

(a) **(6p)** Find the general solution of the differential equation $y'' - 12y' + 20y = 3e^{-t}$.
(b) **(6p)** Find the general solution of the following system of differential equations:

$$\begin{aligned} y_1' &= 3y_1 + 4y_2 \\ y_2' &= 4y_1 - 3y_2 \end{aligned}$$

(c) **(6p)** Find the equilibrium states of the autonomous differential equation $y' = 0.15y(1-y/200)$ and determine their stability. Are any of the equilibrium states globally asymptotically stable?

Question 3.

We consider the function $f(x, y, z) = 16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2 + 10z$.

(a) **(6p)** Find all stationary points of f with $x = 1$.
(b) **(6p)** Show that f has a global maximum point, and find the maximal value of f.
(c) **(6p)** Use the envelope theorem to estimate max $(16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2 + 11z)$.

Question 4.

We consider the Kuhn-Tucker problem

$$\max\ f(x, y, z) = 3x^2 - y^2 - 2z^2 \text{ subject to } 2x^4 + 2y^4 + z^4 \leq 18$$

(a) **(6p)** Write down the Kuhn-Tucker conditions for this problem.
(b) **(6p)** Find all points $(x, y, z; \lambda)$ that satisfy the Kuhn-Tucker conditions.
(c) **(6p)** Show that the best candidate points from (b) are the maximum points, and use this to determine the maximum value.

Question 5.

Extra credit (6p) Solve the logistic differential equation $y' = 0.15y(1 - y/200)$, and determine the time it takes for the system to reach 90% of the carrying capacity when $y_0 = 50$.